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MERIDIAN ELLIPSE

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The x and z coordinates of a point on the meridian ellipse can be calculated given the reduced, geodetic or geocentric latitudes. Using the reduced latitude, β, (see Figure 1) the formula are derived in the following manner.

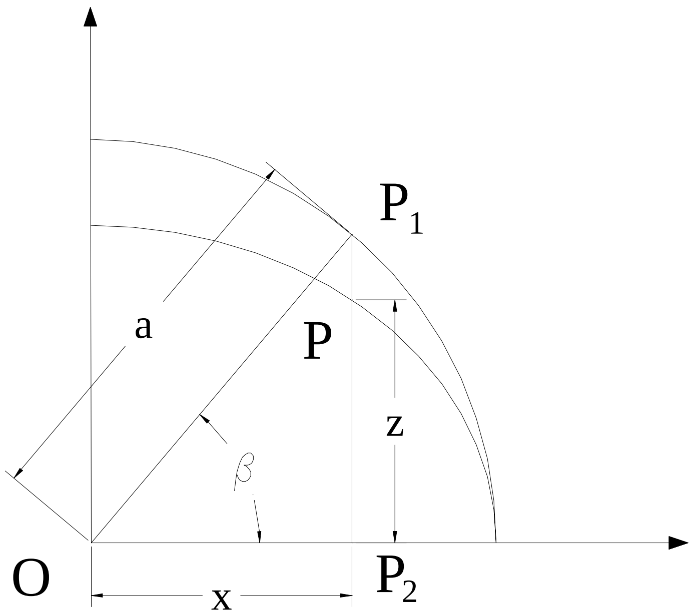

Figure 1. Meridian ellipse using the reduced latitude.

From geometry one can see the triangle OP_1P_2 that the application of the Pythagorean theorem results in

$$(OP_2)^2 + (P_2P_1)^2 = a^2 \tag{1}$$

The equation of the ellipse can be shown to be

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{2}$$

Since x = OP_2 and z = P_2P, substitute these relationships into (2) results in

$$\frac{(OP_2)^2}{a^2} + \frac{(P_2P)}{b^2} = 1 \tag{3}$$

which can be rewritten in terms of the semi-major axis by multiplying both sides of the equation with a^2.

$$(OP_2)^2 + (P_2P)^2 \frac{a^2}{b^2} = a^2 \tag{3a}$$

Combining (1) with (3a) results in

$$(OP_2)^2 + (P_2P)^2 \frac{a^2}{b^2} = a^2 = (OP_2)^2 + (P_2P_1)^2 \tag{4}$$

Solve for P_2P

$$(P_2P_1)^2 = (P_2P)^2 \frac{a^2}{b^2}$$

$$(P_2P)^2 = \frac{b^2}{a^2}(P_2P_1)^2$$

$$P_2P = \frac{b}{a}(P_2P_1) \tag{5}$$

From Figure 1, one can see that $P_2P_1 = a \sin \beta$. Therefore,

$$z = P_2P = \frac{b}{a} P_2P_1 = \frac{b}{a} a \sin \beta$$

or

$$\boxed{z = b \sin \beta} \tag{6}$$

and

$$\boxed{x = a \cos \beta} \tag{7}$$

Example: If $\beta = 43^{o}\ 37'\ 25''$ and using the GRS 80 ellipsoid, the meridional coordinates are found from:

$$\begin{aligned} x &= a \cos\beta \\ &= (6{,}378{,}137\text{m}) \cos 43^{o} 37' 25'' \\ &= 4{,}617{,}054.3716\text{m} \end{aligned}$$

$$\begin{aligned} z &= b\sin\beta \\ &= (6{,}356{,}752.3142\text{m})\sin 43^{o}37'25'' \\ &= 4{,}385{,}637.2730\text{m} \end{aligned}$$

Intuitively, one should be able to look at equations (6) and (7) to see if they look "correct". Let's take an example. If the latitude is 0^o then we see that x = a and z = 0, which makes sense. In a similar vein, one can look at the example and say that it looks "correct" in its presentation. Since the latitude is less tan 45^o and a>b then we know that x must be larger than z. Above 45^o, z will begin to become larger than x.

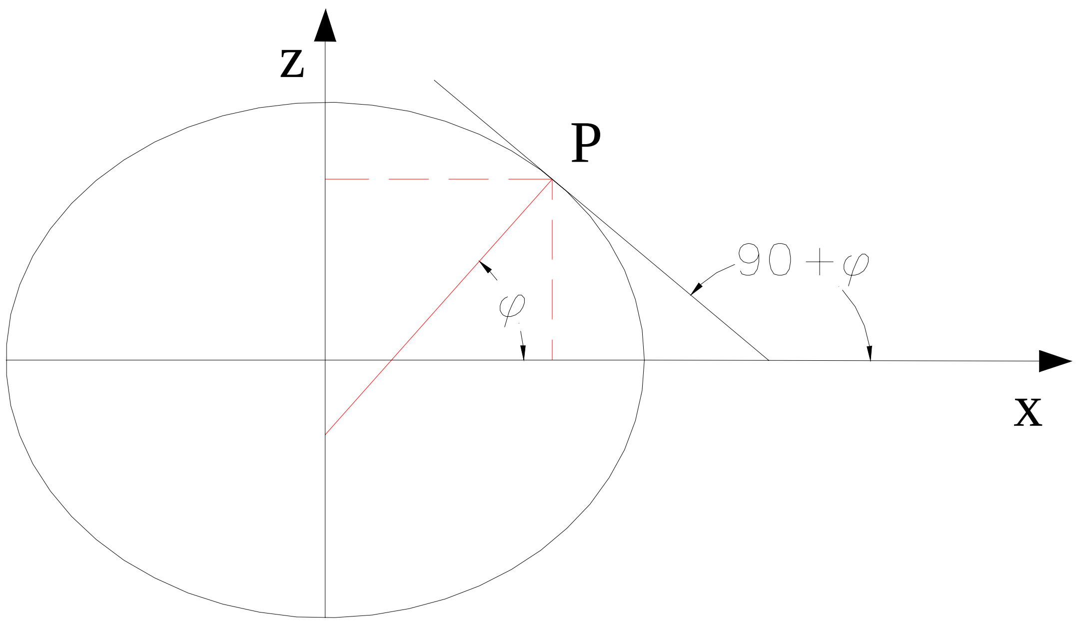

Figure 2. Meridian ellipse using the geodetic latitude.

The determination of the x and z coordinates using the geodetic latitude is shown next. Looking at Figure 2, the slope of the tangent line $\left(dx/dz\right)$ is given as

$$\frac{dy}{dx} = \tan\left(90^o + \varphi\right) = -\cot\varphi = -\frac{\cos\varphi}{\sin\varphi} \tag{8}$$

Rewrite the equation of ellipse (2) into the following form

$$b^2x^2 + a^2z^2 = a^2b^2 \tag{9}$$

Rearranging,

$$z^2 = \frac{a^2b^2 - b^2x^2}{a^2} \quad \Rightarrow \quad z = \sqrt{\frac{a^2b^2 - b^2x^2}{a^2}} \tag{10}$$

Taking the derivative with respect to x will give

$$\frac{dz}{dx} = \frac{1}{2\sqrt{\frac{a^2b^2 - b^2x^2}{a^2}}} \cdot \left(\frac{-b^2}{a^2}\right) 2x = -\frac{b^2}{a^2}\frac{x}{z} = -\frac{\cos\varphi}{\sin\varphi} \tag{11}$$

One can rearrange (11) into

$$b^2 x \sin\varphi = a^2 z \cos\varphi$$

Square both sides yields

$$b^4x^2 \sin^2\varphi - a^4z^2\cos^2\varphi = 0 \tag{12}$$

Multiply (9) by $-b^2\sin^2\varphi$ will result in the next equation

$$-b^4x^2\sin^2\varphi - a^2b^2z^2\sin^2\varphi + a^2b^4\sin^2\varphi = 0 \tag{13}$$

Adding (12) and (13) then multiplying by -1 results in

$$a^4z^2\cos^2\varphi + a^2b^2\sin^2\varphi - a^2b^4\sin^2\varphi = 0 \tag{14}$$

Solve for z

$$z^2\left(a^4\cos^2\varphi + a^2b^2\sin^2\varphi\right) = a^2b^4\sin^2\varphi$$

$$z^2 = \frac{a^2b^4\sin^2\varphi}{a^2\left(a^2\cos^2\varphi + b^2\sin^2\varphi\right)} = \frac{b^4\sin^2\varphi}{a^2\cos^2\varphi + b^2\sin^2\varphi}$$

$$\boxed{z = \frac{b^2\sin\varphi}{\left[a^2\cos^2\varphi + b^2\sin^2\varphi\right]^{1/2}}} \tag{15}$$

The x coordinate can be found using a similar method of reduction yielding

$$x = \frac{a^2 \cos\varphi}{\left[a^2 \cos^2\varphi + b^2 \sin^2\varphi\right]^{1/2}} \tag{16}$$

But, recall

$$e^2 = \frac{a^2 - b^2}{a^2} \Rightarrow b^2 = a^2\left(1 - e^2\right) \tag{17}$$

Substitute (17) into the denominator of (15) and (16) gives

$$\left[a^2 \cos^2\varphi + b^2 \sin^2\varphi\right]^{1/2} = \left[a^2 \cos^2\varphi + a^2 \sin^2\varphi - a^2 e^2 \sin^2\varphi\right]^{1/2}$$

$$= \left[a^2\left(\cos^2\varphi + \sin^2\varphi - e^2 \sin^2\varphi\right)\right]^{1/2}$$

$$= a\left[1 - e^2 \sin^2\varphi\right]^{1/2}$$

Upon substituting this relationship back onto the denominator of (15) and (16) yields:

$$x = \frac{a\cos\varphi}{\left(1 - e^2 \sin^2\varphi\right)^{1/2}} \tag{18}$$

$$z = \frac{a\left(1 - e^2\right)\sin\varphi}{\left(1 - e^2 \sin^2\varphi\right)^{1/2}} \tag{19}$$

Example: If φ = 43° 43’ 11” and again using the GRS 80 ellipse, the meridional coordinates x and z are found by:

$$\begin{aligned} x &= \frac{a\cos\varphi}{\left(1 - e^2 \sin^2\varphi\right)^{1/2}} \\ &= \frac{(6{,}378{,}137\text{m})\cos 43^\circ 43' 11''}{\left(1 - 0.0066943800229\ \sin 43^\circ 43' 11''\right)} \\ &= 4{,}617{,}054.2002\text{m} \end{aligned}$$

$$z = \frac{a\left(1-e^2\right)\sin\varphi}{\left(1-e^2\sin^2\varphi\right)^{1/2}}$$

$$= \frac{(6{,}378{,}137m)(1-0.0066943800229)\sin 43^o 43'11''}{\left(1-0.0066943800229\ \sin 43^o 43'11''\right)}$$

$$= 4{,}385.637.4523m$$

You will notice that these two values are very close to the meridional coordinates found using the reduced latitude. This is not by change. The reduced latitude, and for that the geocentric latitude that follows, were computed based on a given geodetic latitude. The discrepancy is due to rounding off the latitude to the nearest second of arc. Otherwise, they would be exactly the same.

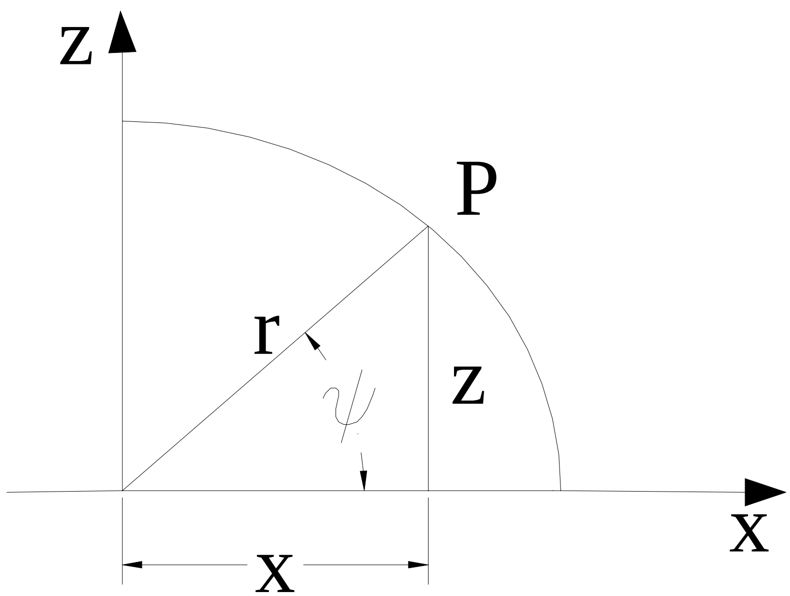

Figure 3. Meridian ellipse using the geocentric latitude.

The determination of the x and z meridional coordinates using the geocentric latitude is developed as follows. From Figure 3, one can write

$$x = r\ \cos\psi \tag{20}$$

$$z = r\ \sin\psi \tag{21}$$

where: r = the geocentric radius, and
ψ = the geocentric latitude

Substitute (20) and (21) into (2) results in

$$\frac{r^2\cos^2\psi}{a^2}+\frac{r^2\sin^2\psi}{b^2}=1 \tag{22}$$

Solving for r:

$$r^2\left(\frac{b^2\cos^2\psi+a^2\sin^2\psi}{a^2b^2}\right)=1$$

$$r^2=\frac{a^2b^2}{b^2\cos^2\psi+a^2\sin^2\psi} \tag{23}$$

Substitute (17), which was written as $b^2=a^2\left(1-e^2\right)$ into (23)

$$r^2=\frac{a^2a^2\left(1-e^2\right)}{a^2\left(1-e^2\right)\cos^2\psi+a^2\sin^2\psi}$$

$$=\frac{a^2\left(1-e^2\right)}{\cos^2\psi-e^2\cos^2\psi+\sin^2\psi}$$

$$\boxed{r=\frac{a\left(1-e^2\right)^{1/2}}{\sqrt{1-e^2\cos^2\psi}}} \tag{24}$$

Substituting (24) into (20) and (21) results in the equations to compute the meridional coordinates using the geocentric latitude.

$$\boxed{x=\frac{a\left(1-e^2\right)^{1/2}\cos\psi}{\sqrt{1-e^2\cos^2\psi}}} \tag{25}$$

$$\boxed{z=\frac{a\left(1-e^2\right)^{1/2}\sin\psi}{\sqrt{1-e^2\cos^2\psi}}} \tag{26}$$

Example: Given ψ = 43^{o} 31’ 39” and the GRS 80 ellipse, the meridional coordinates are:

$$\begin{aligned} x &= \frac{a\left(1-e^2\right)^{1/2}\cos\psi}{\sqrt{1-e^2\cos^2\psi}} \\ &= \frac{6{,}378{,}138\text{m}\left(1-0.0066943800229\right)^{1/2}\cos 43^\circ 31'39''}{\sqrt{1-0.0066943800229\ \cos 43^\circ 31'39''}} \\ &= 4{,}617{,}055.7337\text{m} \end{aligned}$$

$$\begin{aligned} z &= \frac{a\left(1-e^2\right)^{1/2}\sin\psi}{\sqrt{1-e^2\cos^2\psi}} \\ &= \frac{6{,}378{,}138\text{m}\left(1-0.0066943800229\right)^{1/2}\cos 43^\circ 31'39''}{\sqrt{1-0.0066943800229\ \cos 43^\circ 31'39''}} \\ &= 4{,}385{,}635.8487\text{m} \end{aligned}$$

Some useful relationships between the three different latitudes can now be developed. For example, using the geometry in Figure 3, write

$$\tan\psi = z/x$$

Substituting the ratios $(6)/(7)$ and $(19)/(18)$ into this last relationship will yield

$$\tan\psi = \frac{b}{a}\tan\beta = \left(1-e^2\right)\tan\varphi$$

From this basic equation, one can write the following basic relationships between the latitudes:

$$\tan\psi = \left(1-e^2\right)^{1/2}\tan\beta = \left(1-e^2\right)\tan\varphi \tag{27}$$

$$\tan\beta = \left(1-e^2\right)^{1/2}\tan\varphi \tag{28}$$

$$\tan\varphi = \left(1+e'^2\right)^{1/2}\tan\beta \tag{29}$$

MERIDIONAL ELLIPSE

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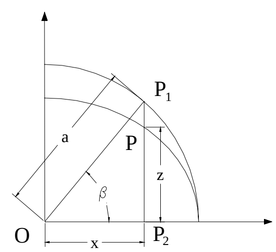
Meridian ellipse using the reduced latitude

$$z = b \sin\beta$$

$$x = a \cos\beta$$

where a and b are the semi-major and semi-minor axes of the ellipse and β is the reduced latitude



Meridian ellipse using the reduced latitude

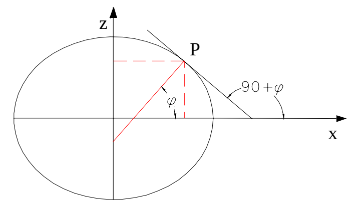
- *Example: using GRS 80 and*

$\beta = 43^o\ 37'\ 25''$
Find the x and z meridional coordinates:

$$x = (6{,}378{,}137\text{m}) \cos 43^o 37' 25''$$
$$= 4{,}617{,}054.3716\text{m}$$
$$z = (6{,}356{,}752.3141\text{m}) \sin 43^o 37' 25''$$
$$= 4{,}385{,}637.2730\text{m}$$

Meridian ellipse using the geodetic latitude

- *Begin by writing equation of ellipse*
- *Differentiate with respect to x and z (the meridional coordinates)*
  - *Finds slope of ellipse at a point*



Meridian ellipse using the geodetic latitude

- *Meridional coordinates found by:*

$$x = \frac{a\cos\varphi}{\left(1-e^2\sin^2\varphi\right)^{1/2}}$$

$$z = \frac{a\left(1-e^2\right)\sin\varphi}{\left(1-e^2\sin^2\varphi\right)^{1/2}}$$

Meridian ellipse using the geodetic latitude

- *Example:* $\varphi = 43^\circ 43'11''$

Find the meridional coordinates:

$$x = \frac{a\cos\phi}{\left(1-e^2\sin^2\varphi\right)^{1/2}} = \frac{6{,}378{,}137\cos 43^\circ 43'11''}{\left(1-0.00669438002\sin^2 43^\circ 43'11''\right)}$$

$$= 4{,}617{,}054.2002\text{m}$$

$$z = \frac{a\left(1-e^2\right)\sin\phi}{\left(1-e^2\sin^2\varphi\right)^{1/2}} = \frac{6{,}378{,}137\left(1-0.00669438002\right)\sin 43^\circ 43'11''}{\left(1-0.00669438002\sin^2 43^\circ 43'11''\right)}$$

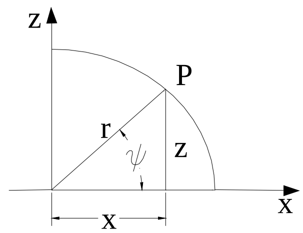
$$= 4{,}385{,}637.4523\text{m}$$

Meridian ellipse using the geocentric latitude

- *From the figure,*

$$x = r\cos\psi$$
$$z = r\sin\psi$$

- *Substitute into equation of ellipse and solve for r*



Meridian ellipse using the geocentric latitude

$$r = \frac{a\left(1-e^2\right)^{1/2}}{\sqrt{1-e^2\cos^2\psi}} \qquad x = \frac{a\left(1-e^2\right)^{1/2}\cos\psi}{\sqrt{1-e^2\cos^2\psi}}$$

- *Substitute into previous equations yields the meridional coordinates*

$$z = \frac{a\left(1-e^2\right)^{1/2}\sin\psi}{\sqrt{1-e^2\cos^2\psi}}$$

Meridian ellipse using the geocentric latitude

- *Example:* ψ = 43° 31' 39"

Find the meridional coordinates:

$$x = \frac{a\left(1-e^2\right)^{1/2}\cos\psi}{\sqrt{1-e^2\cos^2\psi}} = \frac{6{,}378{,}138\text{m}\left(1-0.0066943800229\right)^{1/2}\cos 43^\circ 31'39''}{\sqrt{1-0.0066943800229\cos 43^\circ 31'39''}}$$
$$= 4{,}617{,}055.7337\text{m}$$

$$z = \frac{a\left(1-e^2\right)^{1/2}\sin\psi}{\sqrt{1-e^2\cos^2\psi}} = \frac{6{,}378{,}138\text{m}\left(1-0.0066943800229\right)^{1/2}\cos 43^\circ 31'39''}{\sqrt{1-0.0066943800229\cos 43^\circ 31'39''}}$$
$$= 4{,}385{,}635.8487\text{m}$$

RELATIONSHIPS BETWEEN GEODETIC, REDUCED AND GEOCENTRIC LATITUDES

- *From the last figure,* $\tan\psi = z/x$
- *Substitute the meridional coordinates already computed and after manipulation, results in:*

$$\tan\psi = \left(1-e^2\right)^{1/2}\tan\beta = \left(1-e^2\right)\tan\varphi$$

$$\tan\beta = \left(1-e^2\right)^{1/2}\tan\varphi$$

$$\tan\varphi = \left(1+e'^2\right)^{1/2}\tan\beta$$