2

GEOMETRIC RELATIONSHIPS ON THE SPHERE

Surveying Engineering Department
Ferris State University

Arc Distance on a Sphere

The arc length on a surface of a sphere can be easily computed by multiplying the angle measured at the center of the sphere by the distance. Here the angle is measured in radians. Hence,

$$\widehat{s} = R\gamma$$

where: $\widehat{s}$ = the arc length between two points on the surface of the sphere,
R = the radius of the sphere, and
γ = the angle measured at the center of the sphere subtending the two points on the surface.

On the surface of the earth (considering the earth as a sphere) there are three different types of arc measurements (see figure 1). They are the length of arc on a meridian, length of arc on a parallel, and a length of arc on any great circle.

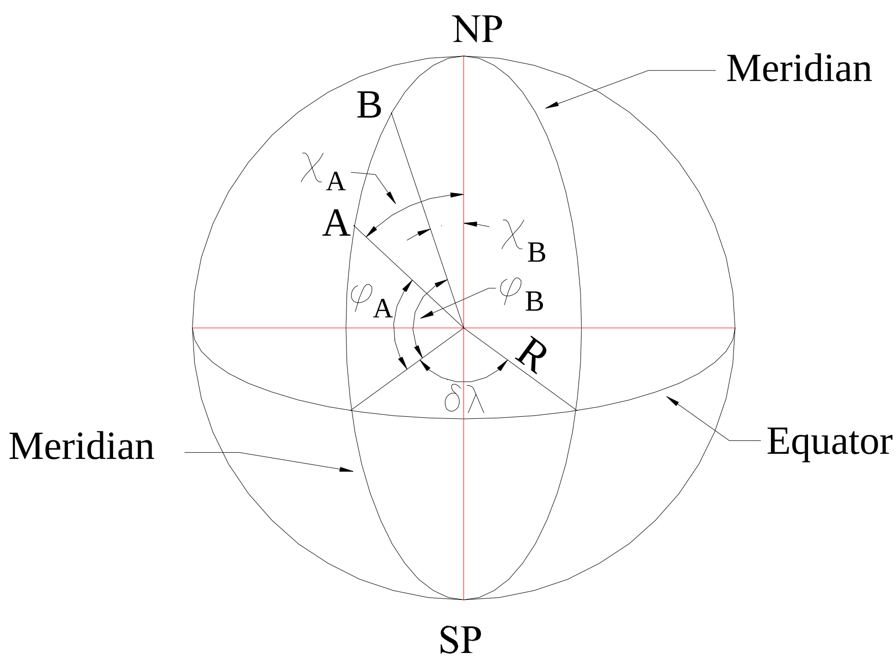


Figure 1. Length of an arc on a meridian.

The length of the arc on a meridian is depicted in figure 1. Using the basic relationship for computing an arc length on the sphere, the arc length from point A to the Equator can be shown to be

$$s_m = R\,\varphi_A$$

where s_m = the arc length along the meridian from the Equator to point A,
R = the radius of the earth (spherical radius), and
φ_A = the latitude of point A.

The arc length can also be computed from the nearest pole (in this case the North Pole) using the co-latitude (χ). Thus,

$$s'_m = R\,\chi_A$$

Here s'_m is used to designate the arc distance from A to the North Pole. The arc distance between points A (φ_A, λ_A) and B (φ_B, λ_B) where both points have the same longitude ($\lambda_A = \lambda_B$) is found in a similar fashion.

$$s''_m = R\,\delta\varphi$$

where s''_m is being used to designate the arc length between two points and $\delta\varphi = \left|\varphi_A - \varphi_B\right|$.

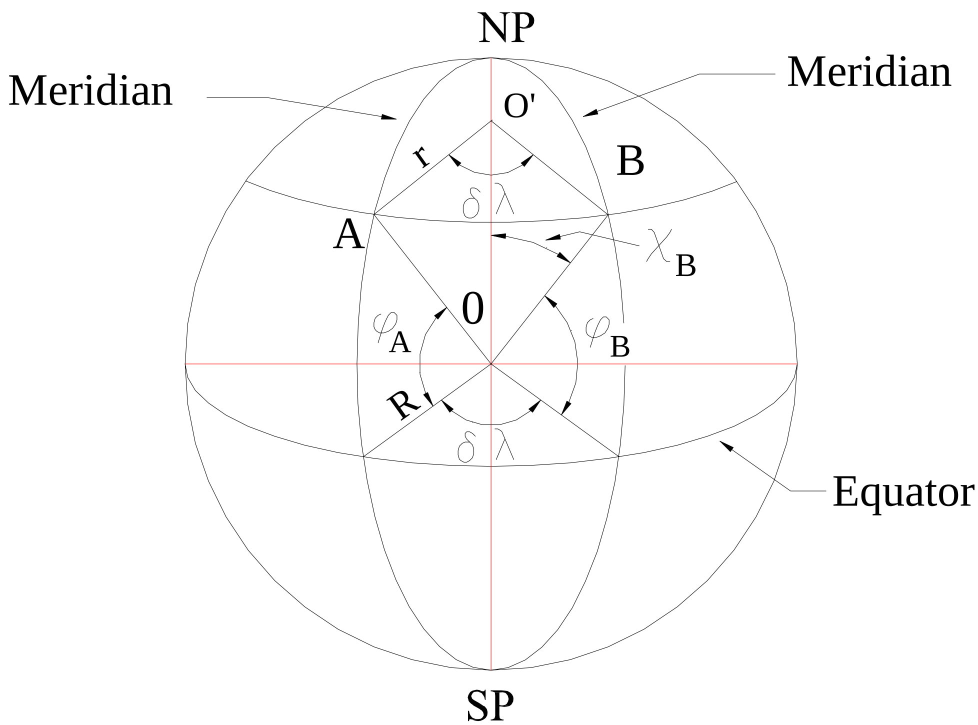


Figure 2. Length of an arc along a parallel.

The computation of the arc length along a parallel can be shown to be similar to that shown for the arc length along a meridian. There are two main differences. First, the difference in longitude ($\delta\lambda$) is used as the angle. Second, the radius of the sphere measured along the rotational axis of the earth changes. At the Equator, the arc distance between the two meridians (arc distance between D and E) shown in figure 2) is found as

$$s_E = R\ \delta\lambda$$

where: s_E is the arc distance along the Equator,
R is the radius of the earth, and
$\delta\lambda$ is the change in longitude, $\delta\lambda = |\lambda_D - \lambda_E|$

Looking at the triangle OO'A, one can see that this is a right triangle. Therefore, the radius, r, from O' to A can be found using either the co-latitude or latitude as:

$$r = R \sin\chi$$
$$r = R \cos\varphi$$

Since the arc distance between A and B can be shown to be equal to $r \cdot \delta\lambda$, the arc distance along the parallel (s_p) can be presented as:

$$s_p = R \cos\varphi\ \delta\lambda$$

It should be evident that this formula is also valid at the Equator. If $\varphi = 0^o$ then $\cos \varphi = 1$ and the arc distance becomes $s_E = R\ \delta\lambda$.

The length of an arc for any great circle on a spherical earth can be computed using spherical trigonometry. The angular distance, z, between A and B is the unknown. The corresponding arc length on the surface is 's'. Points A and B lie on meridional arcs which have a lengths of χ_A and χ_B which are the colatitudes of points A and B respectively (figure 3).

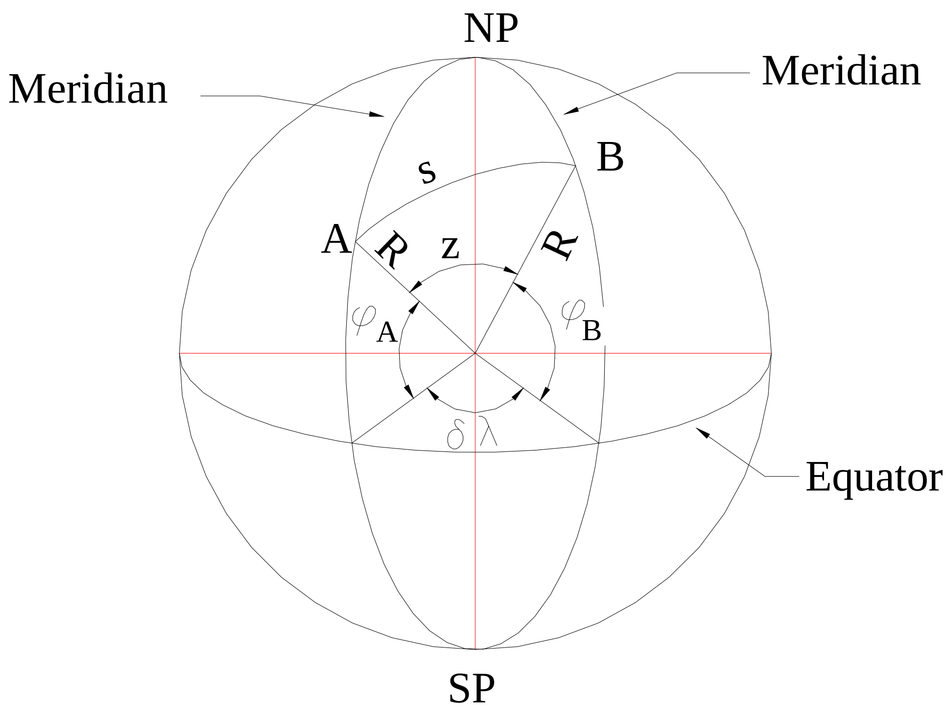


Figure 3. Arc length of any great circle.

Then, from the cosine formula,

$$\cos z = \cos \chi_A \cos \chi_B + \sin \chi_A \sin \chi_B \cos \delta\lambda$$

or expressed in terms of the latitude:

$$\cos z = \sin \phi_A \sin \phi_B + \cos \phi_A \cos \phi_B \cos \delta\lambda$$

and the arc length AB is calculated according to

$$s = Rz$$

There is a problem with this approach though. Snyder[1] points out that it does not work for values close to 0. In other words, it is an ill-conditioned system. Instead, he suggests using the following form:

$$\sin\left(\frac{z}{2}\right) = \left[\sin^2\left(\frac{\phi_B - \phi_A}{2}\right) + \cos\phi_A \cos\phi_B \sin^2\left(\frac{\lambda_B - \lambda_A}{2}\right)\right]^{\frac{1}{2}}$$

[1] Snyder, John. Map Projections – A Working Manual, U.S. Government Printing Office, Washington, D.C.

This formula is exact and very accurate in practice for values of z from 0 to nearly 180°. Then, the arc distance, s, is computed as before.

Area of the Surface

From basic math we know that the area of a circle is

$$A = \pi R^2 = \frac{\pi}{4} D^2$$

From calculus, the general formula for the surface area of a surface of revolution is

$$S = \int 2\pi\rho\, ds$$

where: S is the surface area,
ρ is the distance from the axis of revolution to the surface element, and
ds is the arc length

Figure 4 shows the geometry of the circle. Assume that it is revolved about the x-axis. Given the coordinates of a point on the circle, the radius can be found using the Pythagorean theorem.

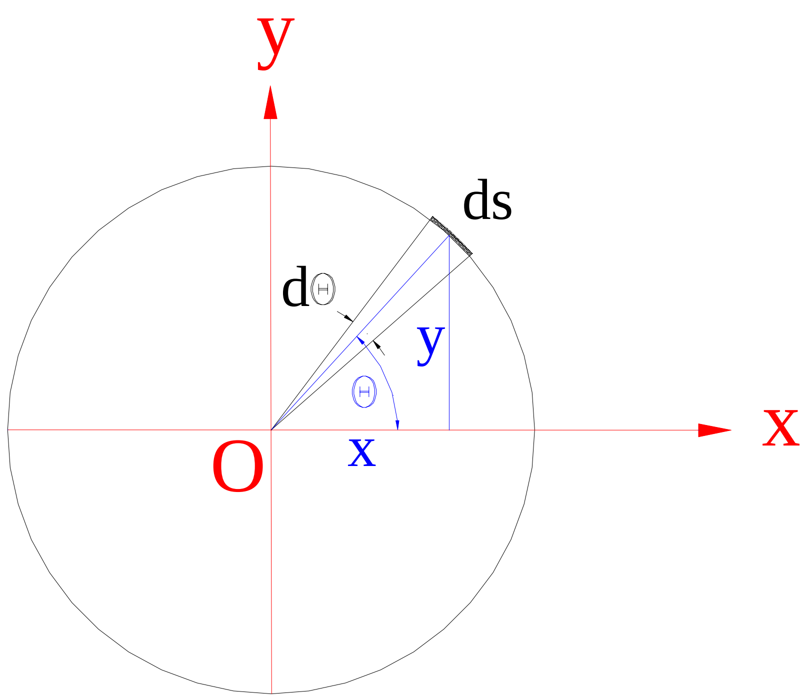


Figure 4. Geometry for area of a surface.

$$r^2 = x^2 + y^2$$

The surface area for a differentially small spot on the sphere can be represented as

$$dS = 2\pi y\, ds$$

The differentially small surface distance, ds, is defined as

$$ds = \sqrt{dx^2 + dy^2}$$

The x and y coordinates of a point on the sphere can be computed using simple trigonometry.

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Differentiating,

$$dx = -r\sin\theta\, d\theta \qquad \text{and} \qquad dy = r\cos\theta\, d\theta$$

It is also clear that

$$ds = r\ d\theta$$

Substituting the values for dy and ds into the general equation for the surface area, we have

$$\begin{aligned} dS &= 2\pi(r\sin\theta) r\,\theta \\ &= 2\pi r^2 \sin\theta\, d\theta \end{aligned}$$

It is fairly evident that if we integrate this equation as θ varies from 0 to π that the upper semicircle surface area is

$$\begin{aligned} S &= \int_0^{\pi} 2\pi r^2 \sin\theta\, d\theta \\ &= 2\pi r^2 [-\cos\theta]_0^{\pi} \\ &= 4\pi r^2 \end{aligned}$$

The general formula for the computation of the surface area of a zone as shown in figure 5 is

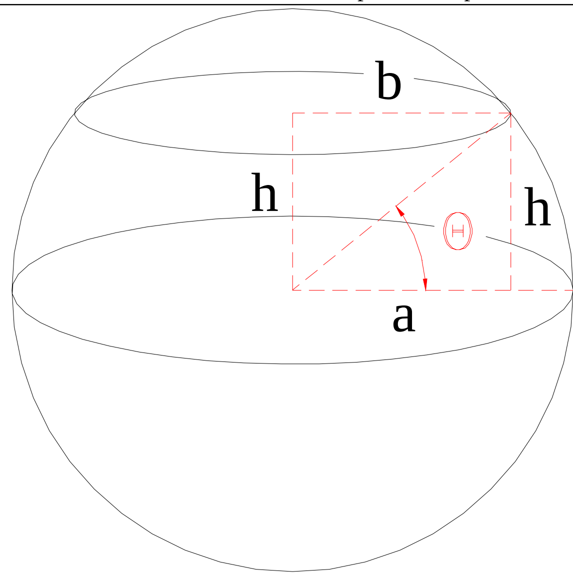


Figure 5. Surface area of a zone.

$$S = 2\pi R h$$

but h can be written

$$h = R \sin\theta$$

therefore,

$$S = 2\pi R^2 \sin\theta$$

If we use the latitude, φ, instead of the angle θ, the general equation for the surface area becomes

$$S = 2\pi R^2 \sin\varphi$$

Defining the zone by φ_1 and φ_2, the surface area of the zone between these two latitudes is the difference in their surface areas.

$$\begin{aligned} S &= 2\pi R^2 \sin\varphi_1 - 2\pi R^2 \sin\varphi_2 \\ &= 2\pi R^2 (\sin\varphi_1 - \sin\varphi_2) \end{aligned}$$

Example: Find the surface area for a portion of the earth defined by 30^o N, 50^o N, 70^o W, and 120^o W. Assume a mean radius of the earth of 6,370,000 m.

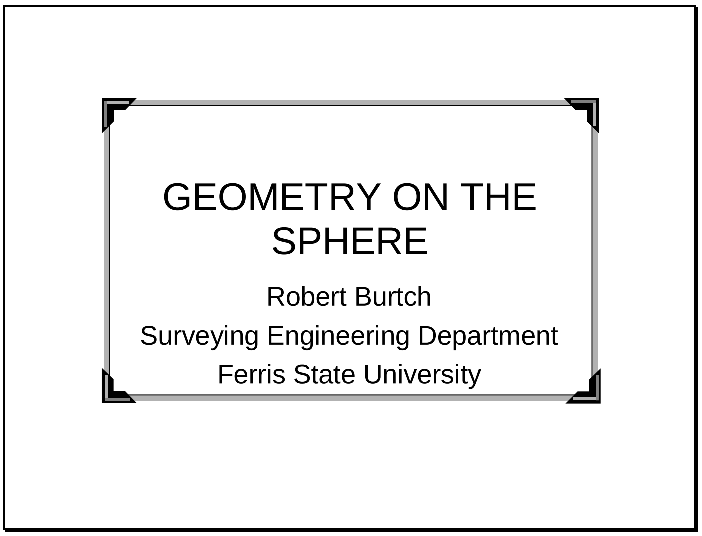
Solution: The surface area of the zone is defined by the latitude as:

$$\begin{aligned} S &= 2\pi R^2 (|\sin\varphi_1 - \sin\varphi_2|) \\ &= 2\pi (6{,}370{,}000)^2 \left|\sin 50^o - \sin 30^o\right| \end{aligned}$$

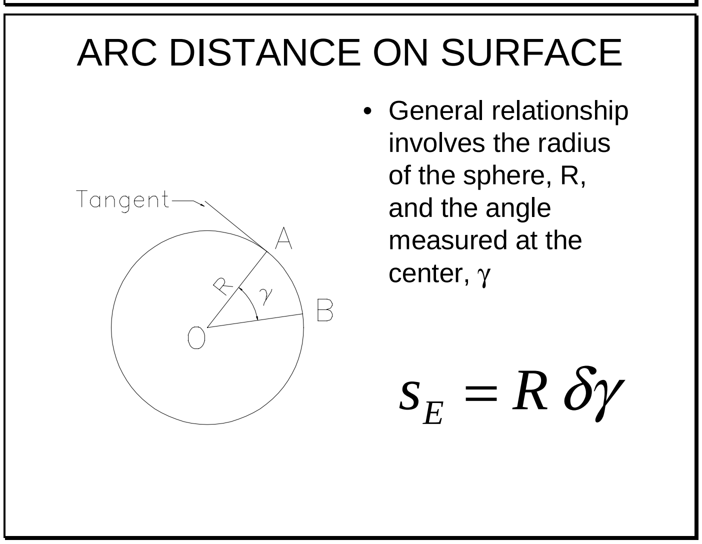
This is the total area between 30^o and 50^o north latitudes around the whole sphere. The desired area is only 50^o wide (120^o W – 70^o W). Therefore,

$$S = \left(\frac{50^o}{360^o}\right) 2\pi (6{,}370{,}000)^2 \left(\sin 50^o - \sin 30^o\right)$$
$$= 9.4206 \text{ x} 10^{12} \text{ m}^2$$
$$= 9{,}420{,}640 \text{ km}^2$$

Slide 1



Slide 2



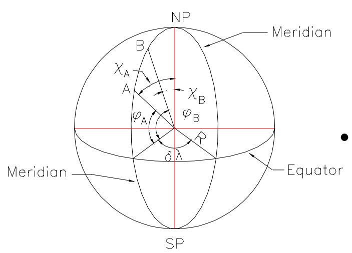
Slide 3

LENGTHS OF ARCS ON EARTH'S SURFACE

- 3 kinds of arc measurements
  - Length of arc on a meridian
  - Length of arc on a parallel
  - Length of arc on any great circle

Slide 4

LENGTH OF ARC ON MERIDIAN



- Arc length from Equator to point A is

$$s_m = R\ \varphi_A$$

- Arc length from North Pole is

$$s'_m = R\ \chi_A$$

Slide 5

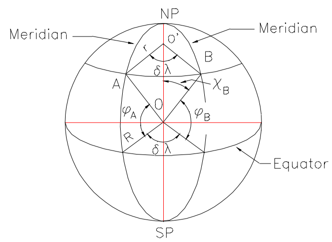
LENGTH OF ARC ON MERIDIAN

- Arc length on the meridian between points A and B
  - Use the difference in latitude, $\delta\varphi = |\varphi_A - \varphi_B|$

$$s''_m = R\ \delta\varphi$$

Slide 6

LENGTH OF ARC ON A PARALLEL



- Triangle OO'A is a right triangle
- Radius, r, computed using either the co-latitude or latitude

$$r = R \sin\chi$$

$$r = R\cos\varphi$$

$$\delta\lambda = |\lambda_D - \lambda_E|$$

Slide 7

LENGTH OF ARC ON A PARALLEL

- Arc length is equal to r·δλ, which is, after substitution for r

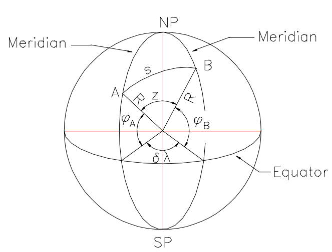
$$s_p = R\cos\varphi\,\delta\lambda$$

- At Equator, $\varphi = 0^o$ and $\cos\varphi = 1$, thus

$$s_E = R\,\delta\lambda$$

Slide 8

LENGTH OF ARC ON ANY GREAT CIRCLE



- Use spherical trigonometry
- Angular distance, z, unknown
- Corresponding arc length is s
- Can use either latitude or co-latitude

Slide 9

LENGTH OF ARC ON ANY GREAT CIRCLE

- From cosine formula

$$\cos z = \cos\chi_A \cos\chi_B + \sin\chi_A \sin\chi_B \cos\delta\lambda$$

or

$$\cos z = \sin\phi_A \sin\phi_B + \cos\phi_A \cos\phi_B \cos\delta\lambda$$

- Arc length computed as

$$s = Rz$$

Slide 10

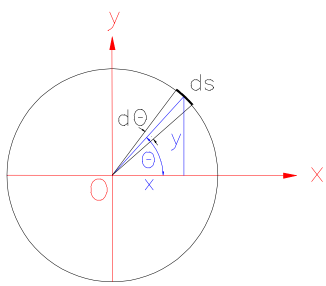
LENGTH OF ARC ON ANY GREAT CIRCLE

- Previous formulas for z are ill-conditioned
  - Do not work for values close to 0
- Alternative form
  - Exact and very accurate

$$\sin\left(\frac{z}{2}\right) = \left[\sin^2\left(\frac{\phi_B - \phi_A}{2}\right) + \cos\phi_A \cos\phi_B \sin^2\left(\frac{\lambda_B - \lambda_A}{2}\right)\right]^{\frac{1}{2}}$$

Slide 11

AREA OF SURFACE



- Area of circle

$$A = \pi R^2 = \frac{\pi}{4} D^2$$

- Surface area of surface of revolution

$$S = \int 2\pi\rho \, ds$$

Slide 12

AREA OF SURFACE

- Surface from differentially small area

$$dS = 2\pi y\, ds$$

- Differentially small surface distance

$$ds = \sqrt{dx^2 + dy^2}$$

Slide 13

AREA OF SURFACE

- Coordinates (x, y) of point on sphere

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

- Differentiating

$$dx = -r\sin\theta\, d\theta \quad \text{and} \quad dy = r\cos\theta\, d\theta$$

- Also:

$$ds = r\; d\theta$$

Slide 14

AREA OF SURFACE

- Substitute dy and ds into general equation for surface area

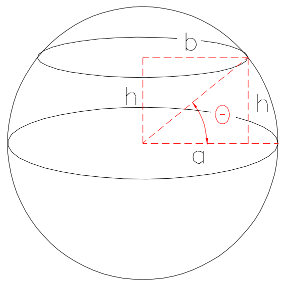
$$dS = 2\pi(r\sin\theta)r\,\theta$$
$$= 2\pi r^2 \sin\theta\, d\theta$$

- Integrating from 0 to π

$$S = \int_0^\pi 2\pi r^2 \sin\theta\, d\theta$$
$$= 2\pi r^2 \left[-\cos\theta\right]_0^\pi$$
$$= 4\pi r^2$$

Slide 15

SURFACE AREA OF ZONE



- General formula

$$S = 2\pi R h$$

- But h can be shown as

$$h = R \sin\theta$$

Slide 16

SURFACE AREA OF ZONE

- Using latitude instead of θ

$$S = 2\pi R^2 \sin\varphi$$

- Defining zone by two latitudes yields

$$\begin{aligned} S &= 2\pi R^2 \sin\varphi_1 - 2\pi R^2 \sin\varphi_2 \\ &= 2\pi R^2 (\sin\varphi_1 - \sin\varphi_2) \end{aligned}$$

Slide 17

EXAMPLE

- Find the surface area for a portion of the earth defined by 30º N, 50º N, 70º W, and 120º W. Assume a mean radius of the earth of 6,370,000 m.

$$\begin{aligned} S &= \left(\frac{50^o}{360^o}\right) 2\pi (6{,}370{,}000)^2 (\sin 50^o - \sin 30^o) \\ &= 9.4206 \text{ x } 10^{12} \text{ m}^2 \\ &= 9{,}420{,}640 \text{ km}^2 \end{aligned}$$