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DIRECT AND INVERSE GEODETIC PROBLEM

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The direct and inverse geodetic problems are one of the most basic questions posed to the geodesist/surveyor. The direct problem can be simply stated as given the latitude and longitude of a beginning point and a distance and azimuth to the second point, compute the latitude and longitude of that second point along with the back azimuth from point 2 to 1. The inverse problem can be formulated as given the latitude and longitude of two points, compute the distance between them along with the forward and reverse azimuths. The problem is complex because the earth is not a plane, or even a sphere. Thus, to solve these problems will require assumptions that limit the accuracy of the results.

Bowring Formulas

Direct Problem

Bowring developed a formulation for the direct problem using a conformal projection of the ellipsoid on the sphere and it is called a Gaussian projection of a second kind. The simplicity of the system lies in that the ellipsoidal geodesic is projected to its corresponding line on a sphere thereby allowing the formulation using spherical trigonometry. Both the direct and inverse solutions are non-iterative. These formulas are valid for lines up to 150 km.

Without derivation, the common equations used by Bowring are:

$$A = \sqrt{\left(1 + e'^2 \cos^4 \varphi_1\right)}$$

$$B = \sqrt{\left(1 + e'^2 \cos^2 \varphi_1\right)}$$

$$C = \sqrt{\left(1 + e'^2\right)}$$

$$w = \frac{A\left(\lambda_2 - \lambda_1\right)}{2}$$

Then the formulas for the direct problem are presented as follows:

$$\sigma = \frac{s\,B^2}{a\,C}$$

$$\lambda_2 = \lambda_1 + \frac{1}{A}\tan^{-1}\left(\frac{A\,\tan\sigma\,\sin\alpha_{12}}{B\,\cos\varphi_1 - \tan\sigma\,\sin\varphi_1\,\cos\alpha_{12}} \right)$$

$$D = \frac{1}{2}\sin^{-1}\left[\sin\sigma\left(\cos\alpha_{12} - \frac{1}{A}\sin\varphi_1\,\sin\alpha_{12}\,\tan w \right)\right]$$

$$\varphi_2 = \varphi_1 + 2D\left[B - \frac{3}{2}e'^2\,D\,\sin\left(2\varphi_1 + \frac{4}{3}BD \right)\right]$$

$$\alpha_{12} = \tan^{-1}\left[\frac{-B\,\sin\alpha_{12}}{\cos\sigma\left(\tan\sigma\,\tan\varphi_1 - B\,\cos\alpha_{12}\right)} \right]$$

Inverse Problem

The inverse problem begins by computing a series of constant values.

$$D = \frac{\Delta\varphi}{2B}\left[1 + \frac{3e'^2}{4B^2}\Delta\varphi\sin\left(2\varphi_1 + \frac{2}{3}\Delta\varphi \right)\right]$$

$$E = \sin D \cos w$$

$$F = \frac{1}{A}\sin w\left(B\cos\varphi_1\cos D - \sin\varphi_1\sin D \right)$$

$$\tan G = F/E$$

$$\sin \sigma/2 = \left(E^2 + F^2 \right)^{1/2}$$

$$\tan H = \left[\frac{1}{A}\left(\sin\varphi_1 + B\cos\varphi_1\,\tan D\right)\tan w \right]$$

Then, the inverse geodetic values, azimuth and distance, are found using the following relationships.

$$\alpha_1 = G - H$$

$$\alpha_2 = G + H \pm 180^o$$

$$s = \frac{a\,C\,\sigma}{B^2}$$

Gauss Mid-Latitude Formula

This approach was first published in English in 1861. This method is based on spherical approximations of the earth. This method is based on using spherical approximations of the earth. Thus, azimuths and distances would be the same on the ellipsoid as they would be on a sphere. This assumption is not true. Thus, this method is best used for distances less than 40 km at latitudes less than 80^o.

Direct Method

Without derivation, the formulas for using the Gauss Mid-Latitude approach are:

$$\Delta\lambda = \frac{s\sin\left(\alpha_{12} + {}^{\Delta\alpha}\!/_{2}\right)}{N_m\cos\varphi_m}$$

$$\Delta\varphi = \frac{s\cos\left(\alpha_{12} + {}^{\Delta\alpha}\!/_{2}\right)}{M_m\cos\left({}^{\Delta\lambda}\!/_{2}\right)}$$

where: $\tan\frac{\Delta\alpha}{2} = \frac{\sin\varphi_m}{\cos\left({}^{\Delta\alpha}\!/_{2}\right)}\tan\frac{\Delta\lambda}{2}$

or $\Delta\alpha = \Delta\lambda\sin\varphi_m\sec\frac{\Delta\varphi}{2} + \frac{\Delta\lambda^3}{12}\left(\sin\varphi_m\sec\frac{\Delta\varphi}{2} - \sec^3\varphi_m\sec^3\frac{\Delta\varphi}{2}\right)$

Because of the nature of these equations, they have to be solved iteratively. The following steps are used in this solution.

1. Solve for the approximate change in latitude ($\Delta\varphi$) by using the measured azimuth instead of $\left(\alpha_{12} + \Delta\alpha/2\right)$ and using the radius of curvature in the meridian at point 1 (M_1) instead of the mean radius of curvature (M_m).
2. Compute the first approximation of the latitude of the second point

$$\varphi_2 = \varphi_1 + \Delta\varphi$$

3. Determine the first approximation of the change in longitude and the longitude of the second point.

$$\lambda_2 = \lambda_1 + \Delta\lambda$$

4. Find the first approximation of the change in azimuth.
5. Using these approximations, update the values M, N, φ_m and the other values listed in the first four steps.

Inverse Problem

The Gauss Mid-Latitude formulae for the inverse problem can be developed into direct computation requiring no iteration. It can be presented as follows (without derivation)

$$s = s_1 \left[\frac{s_1/2N_m}{\sin\left(s_1/2N_m\right)} \right]$$

$$\alpha_{12} = \tan^{-1}\left(\frac{X_1}{X_2}\right) - \frac{\Delta\alpha}{2}$$

$$\alpha_{21} = \alpha_{12} + \Delta\alpha \pm 180^{\circ}$$

where: $s_1 = \left(X_1^2 + X_2^2\right)^{1/2}$. Then,

$$X_1 = s_1 \sin\left(\alpha_{12} + \frac{\Delta\alpha}{2}\right) = N_m \Delta\lambda' \cos\varphi_m$$

$$X_2 = s_1 \cos\left(\alpha_{12} + \frac{\Delta\alpha}{2}\right) = M_m \Delta\varphi' \cos\left(\frac{\Delta\lambda}{2}\right)$$

$$\Delta\varphi' = \Delta\varphi \left(\frac{\sin \Delta\varphi/2}{\Delta\varphi/2}\right)$$

$$\Delta\lambda' = \Delta\lambda \left(\frac{\sin \Delta\lambda/2}{\Delta\lambda/2}\right)$$

$$\Delta\alpha = \Delta\lambda \sin\varphi_m \sec\left(\frac{\Delta\varphi}{2}\right) + F\Delta\lambda^3$$

$$F = \frac{1}{12}\sin\varphi_m \cos^2\varphi_m$$

This approach has an accuracy of about 1 ppm for lines up to 100 km in length.

PUISSANT FORMULAS

The Puissant formulas are not generally used for lines greater than 100 km long. The formulation of this method is based on defining spheres passing through the first point.

Direct Problem

Without derivation, the difference in the latitude can be presented as:

$$\Delta\varphi = s\cos\alpha_{12}B - s^2\sin^2\alpha_{12}C - h\,s^2\sin^2\alpha_{12}E - (\delta\varphi)^2 D$$

where:

$$B = \frac{1}{M_1}$$

$$C = \frac{\tan\varphi_1}{2M_1N_1}$$

$$D = \frac{3e^2 \sin\varphi_1 \cos\varphi_1}{2\left(1 - e^2 \sin^2\varphi_1\right)}$$

$$E = \frac{1 + 3\tan^2\varphi_1}{6N_1^2}$$

$$h = \frac{s\cos\alpha_{12}}{M_1}$$

The term $\delta\varphi$ can be found either from summing the first three terms in the formula used to compute $\Delta\varphi$ or from the relationship

$$\delta\varphi = \frac{s}{M_1}\cos\alpha_{12} - \frac{s^2}{2M_1N_1}\sin^2\alpha_{12}\tan\varphi_1 - \frac{s^3}{6M_1N_1^2}\sin^2\alpha_{12}\left(1 + 3\tan^2\varphi_1\right)$$

The latitude is then found

$$\varphi_2 = \varphi_1 + \Delta\varphi$$

The difference in longitude is found from the follwing equation.

$$\sin\Delta\lambda = \sin\left({}^{s}\!/_{N_2}\right)\sin\alpha_{12}\sec\varphi_2$$

Then the longitude of the second point is found by adding this change in longitude to the initial longitude.

$$\lambda_2 = \lambda_1 + \Delta\lambda$$

The back azimuth is found by

$$\Delta\alpha = \Delta\lambda\sin\varphi_m\sec\left({}^{\Delta\varphi}\!/_{2}\right) + \frac{\Delta\lambda^3}{12}\left[\sin\varphi_m\sec\left({}^{\Delta\varphi}\!/_{2}\right) - \sin^3\varphi_m\sec^3\left({}^{\Delta\varphi}\!/_{2}\right)\right]$$

For short lines up to about 19 km, the Puissant equations can be simplified into the following form.

$$\Delta\varphi = s\cos\alpha_{12}B - s^2\sin^2\alpha_{12}C - (\delta\varphi)^2 D$$

$$\Delta\lambda = \frac{s}{N_2}\sin\alpha_{12}\sec\varphi_2$$

$$\Delta\alpha = \Delta\lambda\sin\varphi_m$$

Inverse Problem

The inverse problem is an iterative solution. First, compute

$$s\sin\alpha_{12} = \frac{N_2\Delta\lambda\cos\varphi_2}{1-\frac{s^2}{6N_2^2}\left(1-\sin^2\alpha_{12}\sec^2\varphi_m\right)}$$

For the first initial approximtion, the numerator is set to 1. Then, solve for

$$s\cos\alpha_{12} = \frac{1}{B}\left[\Delta\varphi + s^2\sin^2\alpha_{12}C + hEs^2\sin^2\alpha_{12} + (\delta\varphi)^2 D\right]$$

Even though h is not known on the right hand side of the equation, one can obtain an estimate for $s\cos\alpha_{12}$. The value for h can be computed in subsequent iterations of these equations. Next, solve for α_{12} using

$$\tan\alpha_{12} = \frac{s\sin\alpha_{12}}{s\cos\alpha_{12}}$$

then

$$s = \left[(s\sin\alpha_{12})^2 + (s\cos\alpha_{12})^2\right]^{1/2}$$

Iterate to a solution and solve for the back azimuth using the same relationship given in the solution for the direct problem.