WHEN PRINCIPAL LINES COINCIDE WITH PARAMETRIC LINES, THAT IS WHEN $a = h$ AND $b = k$, MAXIMUM & MINIMUM DISTORTION IS ALONG THE MERIDIAN AND PARALLEL.

TAKE THE FIRST 2 BASIC EQUATIONS, SQUARE & ADD THEM

$$h^2 = a^2\cos^2 u + b^2 \sin^2 u = a^2(1-\sin^2 u) + b^2 \sin^2 u$$

$$\sin^2 u (a^2 - b^2) = a^2 - h^2$$

FROM WHICH

$$\sin u = \sqrt{\frac{a^2-h^2}{a^2-b^2}}$$

FOR THE $\cos u$ TERM,

$$h^2 = a^2\cos^2 u + b^2(1-\cos^2 u)$$

$$h^2 - b^2 = \cos^2 u (a^2 - b^2)$$

$$\cos u = \sqrt{\frac{h^2-b^2}{a^2-b^2}}$$

DIVIDING $\sin u$ by $\cos u$,

$$\tan u = \sqrt{\frac{a^2-h^2}{k^2-b^2}}$$

IN A SIMILAR FASHION, USING THE 3RD & 4TH BASIC EQUATIONS GIVE.

$$\sin v = \sqrt{\frac{a^2-k^2}{a^2-b^2}} \qquad \cos v = \sqrt{\frac{k^2-b^2}{a^2-b^2}}$$

$$\tan v = \sqrt{\frac{a^2-k^2}{k^2-b^2}}$$

FINALLY, AS A REVIEW:

- THE CONDITION OF EQUIDISTANCE IS : $\ell = 1$
- THE CONDITION OF CONFORMALITY IS : $a = b$
- THE CONDITION OF EQUAL AREA IS : $ab = 1$

SUMMARY OF STEPS

1) CALCULATE LENGTH DISTORTION ALONG MERIDIAN (h) & PARALLEL (k) AND ANGLE BETWEEN PARAMETRIC LINES ON DATUM & PROJECTION SURFACES

$$h = \sqrt{\frac{E'}{E}} \qquad k = \sqrt{\frac{G'}{G}} \qquad \cos\omega = \frac{F}{\sqrt{EG}}$$

$$\cos\omega' = \frac{F'}{\sqrt{E'G'}}$$

2) CALCULATE SEMI-MAJOR & SEMI-MINOR AXES of ELLIPSE FOR THE TISSOT INDICATRIX from

$$(a+b)^2 \sin^2\omega = h^2 + k^2 - 2hk\cos(\omega+\omega')$$

$$(a-b)^2 \sin^2\omega = h^2 + k^2 - 2hk\cos(\omega-\omega')$$

3) CALCULATE the PRINCIPAL DIRECTION ON the DATUM FROM

$$\tan u = \sqrt{\frac{a^2-h^2}{k^2-b^2}} \qquad \sin v = \sqrt{\frac{a^2-k^2}{a^2-b^2}}$$

$$\cos v = \sqrt{\frac{k^2-b^2}{a^2-b^2}} \qquad \tan v = \sqrt{\frac{a^2-k^2}{k^2-b^2}}$$

4) FIND the DISTORTED DIRECTION of the ARBITRARY LINE

$$\tan u' = \frac{b}{a}\tan u$$

5) DETERMINE the LENGTH DISTORTION, l

$$l = \sqrt{a^2\cos^2 u + b^2 \sin^2 u}$$

6) CALCULATE the AREA RATIO

$$\text{area ratio} = ab$$

7) FIND the MAXIMUM DIRECTION DISTORTION

$$\sin\Omega = \frac{a-b}{a+b}$$

EXAMPLE FROM D. L. MALING, COORDINATE SYSTEMS AND MAP PROJECTIONS, 2ND ED., PERGAMON PRESS, OXFORD, UK.

$\varphi = 60^\circ N, \ \lambda = 60^\circ E$

ASSUMING DATUM SURFACE OF SPHERE ($R=1$), then the GAUSSIAN FUNDAMENTAL QUANTITIES FOR the DATUM SURFACE IS:

$$E = R^2 = 1$$

$$F = \phi$$

$$G = R^2 \cos^2\varphi = 0.25$$

MALING GIVES the EQUATIONS FOR the HAMMER-AITOFF PROJECTION AS:

$$x = 2\sqrt{2}\left(\frac{\cos\varphi \sin \lambda/2}{\sqrt{1+\cos\varphi\cos\lambda/2}}\right)$$

$$y = \frac{\sqrt{2}\sin\varphi}{\sqrt{1+\cos\varphi\cos\lambda/2}}$$

the PARTIAL DERIVATIVES w.r.t. φ & λ ARE:

$$\frac{\partial x}{\partial \varphi} = -\sqrt{2}\left[\frac{\sin\varphi\sin\lambda/2\,(2+\cos\varphi\cos\lambda/2)}{(1+\cos\varphi\cos\lambda/2)^{3/2}}\right] = -0.868530$$

$$\frac{\partial y}{\partial \varphi} = \frac{\cos\varphi\,(2+\cos\varphi\cos\lambda/2)+\cos\lambda/2}{\sqrt{2}\,(1+\cos\varphi\cos\lambda/2)^{3/2}} = 0.858423$$

$$\frac{\partial x}{\partial \lambda} = \frac{\cos\varphi\cos\lambda/2\,(2+\cos\varphi\cos\lambda/2)+\cos^2\varphi}{\sqrt{2}\,(1+\cos\varphi\cos\lambda/2)^{3/2}} = 0.537316$$

$$\frac{\partial y}{\partial \lambda} = \left[\frac{1}{2^{3/2}}\right]\left[\frac{\sin\varphi\cos\varphi\sin\lambda/2}{(1+\cos\varphi\cos\lambda/2)^{3/2}}\right] = 0.044622$$

the GAUSSIAN FUNDAMENTAL QUANTITIES FOR the PROJECTION SURFACE ARE:

$$E' = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 = 1.491236$$

$$F' = \left(\frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda}\right) + \left(\frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda}\right) = -0.428370$$

$$G' = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 = 0.290699$$

NEXT, SOLVE FOR THE DISTORTIONS

$$h = \sqrt{\frac{E'}{E}} = \sqrt{E'} = 1.221162$$

$$k = \sqrt{\frac{G'}{G}} = 1.078331$$

$$\omega' = \cos^{-1}\left(\frac{F'}{\sqrt{E'G'}}\right) = 130° 35' 16.8''$$

$$\omega = 90°$$

$$(a+b)^2 = \frac{h^2 + k^2 - 2hk\cos(\omega + \omega')}{\sin^2\omega} = 4.654032$$

$$(a-b)^2 = \frac{h^2 + k^2 - 2hk\cos(\omega - \omega')}{\sin^2\omega} = 0.654032$$

ADD THE LAST 2 EQUATIONS TOGETHER, THEN

$$2a = \sqrt{4.654032} + \sqrt{0.654032}$$

$$a = 1.483022$$

$$b = \sqrt{4.654032} - a = 0.674290$$

$$\tan u = \sqrt{\frac{a^2 - h^2}{k^2 - b^2}} = 1 \quad \therefore u = 45°$$

$$\tan v = \sqrt{\frac{a^2 - k^2}{k^2 - b^2}} = 1 \quad \therefore v = 45°$$

$$\tan u' = \frac{b}{a}\tan u \Rightarrow u' = 24° 27' 01''$$

$$l = \sqrt{a^2\cos^2 u + b^2 \sin^2 u} = 1.151962$$

AREA RATIO $= ab = 1.0$

$$\sin \Omega = \frac{a-b}{a+b} = 0.374874$$

$$\Omega = 22° 00' 55''$$