The Center
for
Photogrammetric Training

PROJECTIVE EQUATIONS

Surveying Engineering Department
Ferris State University

RCB

PHOTO COORDINATE SYSTEM

- Origin at principal point
- Defined as: $$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} x - x_o \\ y - y_o \\ -f \end{bmatrix}$$
- Translating ground coordinates to photo $$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

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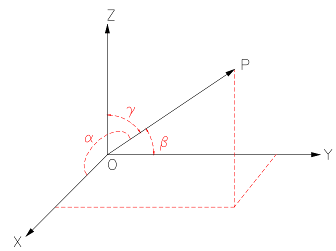
DIRECTION COSINES

- P has coordinates X_P, Y_P, Z_P
- Length of vector OP is

$$OP = \left[X_P^2 + Y_P^2 + Z_P^2\right]^{\frac{1}{2}}$$



- Direction of vector wrt the 3 axes are:

$$\cos\alpha = \frac{X_P}{OP} \qquad \cos\beta = \frac{Y_P}{OP}$$

$$\cos\gamma = \frac{Z_P}{OP}$$

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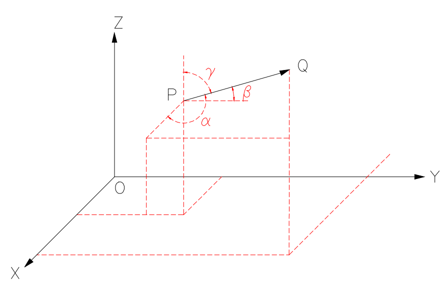
DIRECTION COSINES

- Vector from P to Q defined as

$$\overrightarrow{PQ} = \begin{pmatrix} X_Q - X_P \\ Y_Q - Y_P \\ Z_Q - Z_P \end{pmatrix} = -\overrightarrow{QP}$$



- Length of vector

$$PQ = \left[(X_Q - X_P)^2 + (Y_Q - Y_P)^2 + (Z_Q - Z_P)^2\right]^{1/2}$$

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DIRECTION COSINES

- Direction cosines become:

$$\cos\alpha = \frac{X_Q - X_P}{PQ}$$

$$\cos\beta = \frac{Y_Q - Y_P}{PQ}$$

$$\cos\gamma = \frac{Z_Q - Z_P}{PQ}$$

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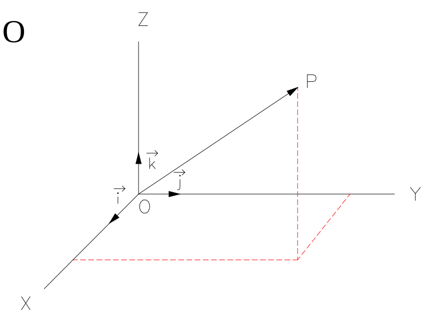
DIRECTION COSINES

- Looking at unit vector, O to P is

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$$



- P has coordinates

  $(x, y, z)^T$

- Given a second set of coordinate axes I, J, K, similar relationships can be formed

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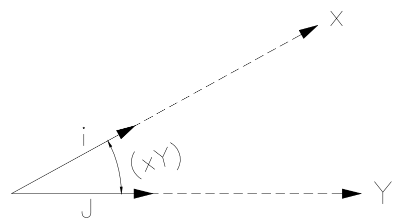
DIRECTION COSINES

- Rotation between Y and x axes
- Writing unit vector in terms of direction cosines:

$$\vec{i} = \begin{pmatrix} \vec{i} \cdot \vec{I} \\ \vec{i} \cdot \vec{J} \\ \vec{i} \cdot \vec{K} \end{pmatrix} = \begin{bmatrix} \cos(xX) \\ \cos(xY) \\ \cos(xZ) \end{bmatrix}$$

- Similarly for $\vec{j}$ and $\vec{k}$



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VECTOR FROM O TO P

$$\overrightarrow{OP} = x \begin{bmatrix} \cos(xX) \\ \cos(xY) \\ \cos(xZ) \end{bmatrix} + y \begin{bmatrix} \cos(yX) \\ \cos(yY) \\ \cos(yZ) \end{bmatrix} + z \begin{bmatrix} \cos(zX) \\ \cos(zY) \\ \cos(zZ) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(xX) & \cos(yX) & \cos(zX) \\ \cos(xY) & \cos(yY) & \cos(zY) \\ \cos(xZ) & \cos(yZ) & \cos(zZ) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\vec{X} = R\ \vec{x}$$

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SEQUENTIAL ROTATIONS

Combination	Axes of Rotation
1) Roll (ω) – Pitch (φ) – Yaw (κ)	x – y - z
2) Pitch (φ) – Roll (ω) – Yaw (κ)	y – x - z
3) Heading (H) – Roll (ω) – Pitch (φ)	z – x - y
4) Heading (H) – Pitch (φ) – Roll (ω)	z – y - x
5) Azimuth (α) – Tilt (t) – Swing (s)	z – x - z
6) Azimuth (α) – Elevation (h) – Swing (s)	z – x - z

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DERIVATION OF GIMBAL ANGLES

- Coordinate transformation shown as

$$\begin{bmatrix} X_P \\ Y_P \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} U_P \\ V_P \end{bmatrix}$$

- Perform a planar rotation of axes in sequence: ω - primary, φ - secondary, κ - tertiary

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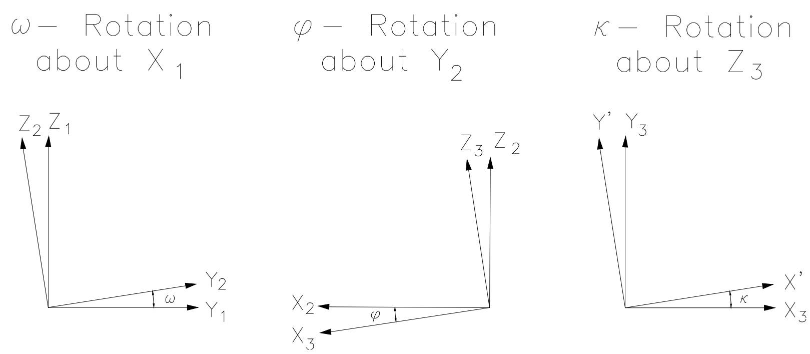
ROTATION ANGLES IN PHOTOGRAMMETRY



ω ROTATION

- In general form:

$$X_2 = X_1 + Y_1 \cdot 0 + Z_1 \cdot 0$$

$$Y_2 = X_1 \cdot 0 + Y_1 \cdot \cos\omega + Z_1 \cdot \sin\omega$$

$$Z_2 = X_1 \cdot 0 + Y_1 \cdot (-\sin\omega) + Z_1 \cdot \cos\omega$$

- In matrix form:

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

- More concisely $C_2 = M_\omega C$

φ ROTATION

- In general form:

$$X_3 = X_2 \cdot \cos\varphi + Y_2 \cdot 0 + Z_2 \cdot (-\sin\varphi)$$
$$Y_3 = X_2 \cdot 0 + Y_2 + Z_2 \cdot 0$$
$$Z_3 = X_2 \cdot \sin\varphi + Y_2 \cdot 0 + Z_2 \cdot \cos\varphi$$

- In matrix form:

$$\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

- More concisely: $C_3 = M_\varphi C_2$

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κ - ROTATION

- In general form:

$$X' = X_3 \cdot \cos\kappa + Y_3 \cdot \sin\kappa + Z_3 \cdot 0$$
$$Y' = X_3 \cdot (-\sin\kappa) + Y_3 \cdot \cos\kappa + Z_3 \cdot 0$$
$$Z' = X_3 \cdot 0 + Y_3 \cdot 0 + Z_3$$

- In matrix form:

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix}$$

- More concisely: $C' = M_\kappa C_3$

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TRANSFORMATION FROM SURVEY PARALLEL SYSTEM

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = M_G \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = M_\kappa M_\varphi M_\omega \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

- M_G becomes, after multiplication

$$M_G = \begin{bmatrix} \cos\varphi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\varphi\cos\kappa & \sin\omega\sin\kappa - \cos\omega\sin\varphi\cos\kappa \\ -\cos\varphi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\varphi\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\varphi\sin\kappa \\ \sin\varphi & -\sin\omega\cos\varphi & \cos\omega\cos\varphi \end{bmatrix}$$

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COMPUTING ROTATION ANGLES

- If rotation matrix known, rotation angles can be computed as shown on the right

$$\tan\omega = \frac{-m_{32}}{m_{33}}$$

$$\sin\phi = m_{31}$$

$$\tan\kappa = \frac{-m_{21}}{m_{11}}$$

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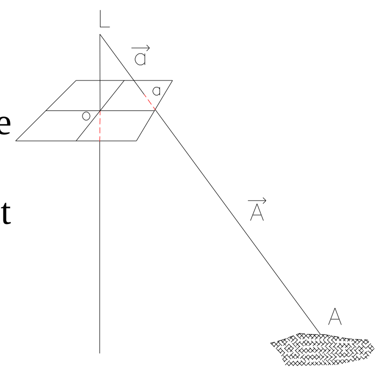
COLLINEARITY CONCEPT

- Line from object space to perspective center is same as line from perspective center to image point
- Relationship shown as:

$$\vec{a} = kM\vec{A}$$



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COLLINEARITY CONCEPT

- Collinearity condition

$$\begin{bmatrix} x - x_o \\ y - y_o \\ -f \end{bmatrix} = k \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

- Ground coordinates are translated to the ground nadir position and rotated to a photo parallel system then scaled to the photograph
  - Predicted photo coordinates

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COLLINEARITY CONCEPT

- Expressing the collinearity concept algebraically:

$$x - x_o = k\left[m_{11}(X - X_L) + m_{12}(Y - Y_L) + m_{13}(Z - Z_L)\right]$$
$$y - y_o = k\left[m_{21}(X - X_L) + m_{22}(Y - Y_L) + m_{23}(Z - Z_L)\right]$$
$$-f = k\left[m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)\right]$$

- Dividing the first 2 equations by the last:

$$x - x_o = -f\left[\frac{m_{11}(X - X_L) + m_{12}(Y - Y_L) + m_{13}(Z - Z_L)}{m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)}\right]$$
$$y - y_o = -f\left[\frac{m_{21}(X - X_L) + m_{22}(Y - Y_L) + m_{23}(Z - Z_L)}{m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)}\right]$$

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COLLINEARITY CONCEPT

- Collinearity equation must satisfy two conditions:

$$m_{11}m_{12} + m_{21}m_{22} + m_{31}m_{32} = 0$$

$$m_{11}^2 + m_{21}^2 + m_{31}^2 = m_{12}^2 + m_{22}^2 + m_{32}^2$$

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COLLINEARITY CONCEPT

- Since $(X - X_L)$, $(Y - Y_L)$ and $(Z - Z_L)$ are proportion to the direction cosines of $\vec{A}$ the collinearity equation can be shown in terms of the direction cosines

$$x - x_o = -f \frac{m_{11}\cos\alpha + m_{12}\cos\beta + m_{13}\cos\gamma}{m_{31}\cos\alpha + m_{32}\cos\beta + m_{33}\cos\gamma}$$

$$y - y_o = -f \frac{m_{21}\cos\alpha + m_{22}\cos\beta + m_{23}\cos\gamma}{m_{31}\cos\alpha + m_{32}\cos\beta + m_{33}\cos\gamma}$$

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COLLINEARITY CONCEPT

- The inverse relationship is

$$X - X_L = (Z - Z_L)\left[\frac{m_{11}(x - x_o) + m_{21}(y - y_o + m_{31}(-f))}{m_{13}(x - x_o) + m_{23}(y - y_o + m_{33}(-f))}\right]$$

$$Y - Y_L = (Z - Z_L)\left[\frac{m_{12}(x - x_o) + m_{22}(y - y_o + m_{32}(-f))}{m_{13}(x - x_o) + m_{23}(y - y_o + m_{33}(-f))}\right]$$

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LINEARIZATION OF COLLINEARITY EQUATION

- For simplicity, let the projective equations be shown in the following form

$$F_1 = (x - x_o) + f\,\frac{U}{W} = 0$$

$$F_2 = (y - y_o) + f\,\frac{V}{W} = 0$$

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LINEARIZATION OF COLLINEARITY EQUATION

- Condition equation shown as:

$$AV + B\Delta + F = 0$$

- Design matrix appears as

$$B = \left[\begin{array}{ccc|cccccc|ccc} \frac{\partial F_1}{\partial x_o} & \frac{\partial F_1}{\partial y_o} & \frac{\partial F_1}{\partial f} & \frac{\partial F_1}{\partial X_L} & \frac{\partial F_1}{\partial Y_L} & \frac{\partial F_1}{\partial Z_L} & \frac{\partial F_1}{\partial \omega} & \frac{\partial F_1}{\partial \varphi} & \frac{\partial F_1}{\partial \kappa} & \frac{\partial F_1}{\partial X_i} & \frac{\partial F_1}{\partial Y_i} & \frac{\partial F_1}{\partial Z_i} \\ \frac{\partial F_2}{\partial x_o} & \frac{\partial F_2}{\partial y_o} & \frac{\partial F_2}{\partial f} & \frac{\partial F_2}{\partial X_L} & \frac{\partial F_2}{\partial Y_L} & \frac{\partial F_2}{\partial Z_L} & \frac{\partial F_2}{\partial \omega} & \frac{\partial F_2}{\partial \varphi} & \frac{\partial F_2}{\partial \kappa} & \frac{\partial F_2}{\partial X_i} & \frac{\partial F_2}{\partial Y_i} & \frac{\partial F_2}{\partial Z_i} \end{array}\right]$$

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LINEARIZATION OF COLLINEARITY EQUATION

- Partial derivatives wrt interior orientation (x_o, y_o, and f only):

$$\frac{\partial F_1}{\partial x_o} = -1 \qquad \frac{\partial F_1}{\partial y_o} = 0 \qquad \frac{\partial F_1}{\partial f} = \frac{U}{W}$$

$$\frac{\partial F_2}{\partial x_o} = 0 \qquad \frac{\partial F_2}{\partial y_o} = -1 \qquad \frac{\partial F_2}{\partial f} = \frac{V}{W}$$

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LINEARIZATION OF COLLINEARITY EQUATION

- Partials taken wrt exposure station

$$\frac{\partial F_1}{\partial P} = f\,\frac{W\left(\frac{\partial U}{\partial P}\right) - U\left(\frac{\partial W}{\partial P}\right)}{W^2} = \frac{f}{W}\left(\frac{\partial V}{\partial P} - \frac{U}{W}\frac{\partial W}{\partial P}\right)$$

$$\frac{\partial F_2}{\partial P} = f\,\frac{W\left(\frac{\partial V}{\partial P}\right) - V\left(\frac{\partial W}{\partial P}\right)}{W^2} = \frac{f}{W}\left(\frac{\partial V}{\partial P} - \frac{V}{W}\frac{\partial W}{\partial P}\right)$$

where P are the parameters

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LINEARIZATION OF COLLINEARITY EQUATION

- For the exposure station coordinates, the partial derivatives of functions U, V, W:

$$\frac{\partial U}{\partial X_L} = -m_{11} \qquad \frac{\partial U}{\partial Y_L} = -m_{12} \qquad \frac{\partial U}{\partial Z_L} = -m_{13}$$

$$\frac{\partial V}{\partial X_L} = -m_{21} \qquad \frac{\partial V}{\partial Y_L} = -m_{22} \qquad \frac{\partial V}{\partial Z_L} = -m_{23}$$

$$\frac{\partial W}{\partial X_L} = -m_{31} \qquad \frac{\partial W}{\partial Y_L} = -m_{32} \qquad \frac{\partial W}{\partial Z_L} = -m_{33}$$

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LINEARIZATION OF COLLINEARITY EQUATION

- Partials of the functions F_1 & F_2 is

$$\frac{\partial F_1}{\partial X_L} = \frac{f}{W}\left(-m_{11} + \frac{U}{W}m_{31}\right) \qquad \frac{\partial F_2}{\partial X_L} = \frac{f}{W}\left(-m_{21} + \frac{V}{W}m_{31}\right)$$

$$\frac{\partial F_1}{\partial Y_L} = \frac{f}{W}\left(-m_{12} + \frac{U}{W}m_{32}\right) \qquad \frac{\partial F_2}{\partial Y_L} = \frac{f}{W}\left(-m_{22} + \frac{V}{W}m_{32}\right)$$

$$\frac{\partial F_1}{\partial Z_L} = \frac{f}{W}\left(-m_{13} + \frac{U}{W}m_{33}\right) \qquad \frac{\partial F_2}{\partial Z_L} = \frac{f}{W}\left(-m_{23} + \frac{V}{W}m_{33}\right)$$

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LINEARIZATION OF COLLINEARITY EQUATION

- Partial of orientation matrix wrt the angles:

$$\frac{\partial M_G}{\partial \omega} = M_\kappa M_\varphi \frac{\partial M_\omega}{\partial \omega} = M_G \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\frac{\partial M_G}{\partial \varphi} = M_\kappa \frac{\partial M_\varphi}{\partial \varphi} M_\omega = M_G \begin{bmatrix} 0 & \sin\omega & -\cos\omega \\ -\sin\omega & 0 & 0 \\ \cos\omega & 0 & 0 \end{bmatrix}$$

$$\frac{\partial M_G}{\partial \kappa} = \frac{\partial M_\kappa}{\partial \kappa} M_\varphi M_\omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} M_G$$

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LINEARIZATION OF COLLINEARITY EQUATION

- Partials wrt the orientation angles:

$$\begin{bmatrix} \frac{\partial U}{\partial \varphi} \\ \frac{\partial V}{\partial \varphi} \\ \frac{\partial W}{\partial \varphi} \end{bmatrix} = \frac{\partial M_G}{\partial \varphi} \begin{bmatrix} X_i - X_L \\ Y_i - Y_L \\ Z_i - Z_L \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial U}{\partial \omega} \\ \frac{\partial V}{\partial \omega} \\ \frac{\partial W}{\partial \omega} \end{bmatrix} = \frac{\partial M_G}{\partial \omega} \begin{bmatrix} X_i - X_L \\ Y_i - Y_L \\ Z_i - Z_L \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial U}{\partial \kappa} \\ \frac{\partial V}{\partial \kappa} \\ \frac{\partial W}{\partial \kappa} \end{bmatrix} = \frac{\partial M_G}{\partial \kappa} \begin{bmatrix} X_i - X_L \\ Y_i - Y_L \\ Z_i - Z_L \end{bmatrix}$$

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LINEARIZATION OF COLLINEARITY EQUATION

- Evaluate the partials of F_1 & F_2 wrt orientation angles:

$$\frac{\partial F_1}{\partial \omega} = \frac{f}{W}\left(\frac{\partial U}{\partial \omega} - \frac{U}{W}\frac{\partial W}{\partial \omega}\right)$$

$$\frac{\partial F_1}{\partial \varphi} = \frac{f}{W}\left(\frac{\partial U}{\partial \varphi} - \frac{U}{W}\frac{\partial W}{\partial \varphi}\right)$$

$$\frac{\partial F_1}{\partial \kappa} = \frac{f}{W}\left(\frac{\partial U}{\partial \kappa} - \frac{U}{W}\frac{\partial W}{\partial \kappa}\right)$$

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LINEARIZATION OF COLLINEARITY EQUATION

- Evaluate the partials of F_1 & F_2 wrt orientation angles:

$$\frac{\partial F_2}{\partial \omega} = \frac{f}{W}\left(\frac{\partial V}{\partial \omega} - \frac{V}{W}\frac{\partial W}{\partial \omega}\right)$$

$$\frac{\partial F_2}{\partial \varphi} = \frac{f}{W}\left(\frac{\partial V}{\partial \varphi} - \frac{V}{W}\frac{\partial W}{\partial \varphi}\right)$$

$$\frac{\partial}{\partial \kappa} = \frac{}{}\left(\frac{\partial}{\partial \kappa} - \frac{}{W}\frac{\partial W}{\partial \kappa}\right)$$

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