3

PROJECTIVE EQUATIONS

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Introduction

In the first section we were introduced to coordinate transformations. The numerical resection problem involves the transformation (rotation and translation) of the ground coordinates to photo coordinates for comparison purposes in the least squares adjustment. Before we begin this process, lets derive the rotation matrix that will be used to form the collinearity condition.

In photogrammetry, the coordinates of the points imaged on the photograph are determined through observations. The next procedure is to compare these photo coordinates with the ground coordinates. On the photograph, the positive x-axis is taken in the direction of flight. For any number of reasons, this will most probably never coincide with the ground X-axis. The origin of the photographic coordinates is at the principal point which can be expressed as

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} x - x_o \\ y - y_o \\ -f \end{bmatrix}$$

where: x, y are the photo coordinates of the imaged point with reference to the intersection of the fiducial axes

x_o, y_o are the coordinates from the intersection of the fiducial axes to the principal point

f is the focal length

Since the origin of the ground coordinates does not coincide with the origin of the photographic coordinate system, a translation is necessary. We can write this as

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

where: X, Y, Z are the ground coordinates of the point

X_L, Y_L, Z_L are the ground coordinates of the ground nadir point

Thus, in the comparison, both ground coordinates and photo coordinates are referenced to the same origin separated only by the flying height. Note that the ground nadir coordinates would correspond to the principal point coordinates in X and Y if the photograph was truly vertical.

Direction Cosines

If we look at figure 1, we can see that point P has coordinates X_P, Y_P, Z_P. The length of the vector (distance) can be defined as

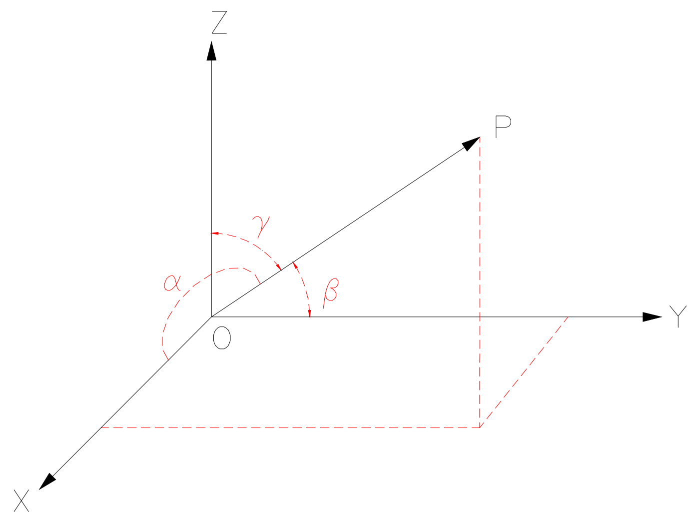

Figure 1. Vector OP in 3-D space.

$$OP = \left[X_P^2 + Y_P^2 + Z_P^2\right]^{\frac{1}{2}}$$

The direction of the vector can be written with respect to the 3 axes as:

$$\cos\alpha = \frac{X_P}{OP}$$
$$\cos\beta = \frac{Y_P}{OP}$$
$$\cos\gamma = \frac{Z_P}{OP}$$

These cosines are called the direction cosines of the vector from O to P. This concept can be extended to any line in space. For example, figure 2 shows the line PQ. Here we can readily see that the vector $\overrightarrow{PQ}$ can be defined as:

$$\overrightarrow{PQ} = \begin{pmatrix} X_Q - X_P \\ Y_Q - Y_P \\ Z_Q - Z_P \end{pmatrix} = -\overrightarrow{QP}$$

The length of the vector becomes

$$PQ = \left[\left(X_Q - X_P\right)^2 + \left(Y_Q - Y_P\right)^2 + \left(Z_Q - Z_P\right)^2\right]^{1/2}$$

and the direction cosines are

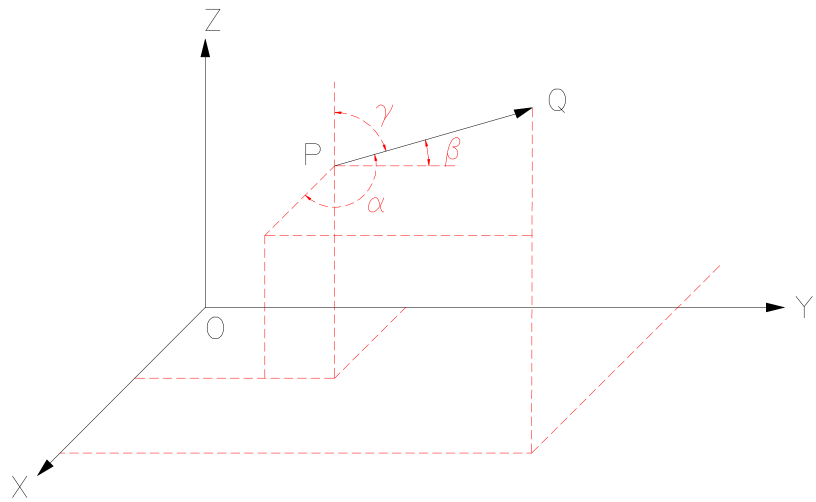

Figure 2. Line vector PQ in space.

$$\cos\alpha = \frac{X_Q - X_P}{PQ}$$

$$\cos\beta = \frac{Y_Q - Y_P}{PQ}$$

$$\cos\gamma = \frac{Z_Q - Z_P}{PQ}$$

If we look at the unit vector as shown in figure 3, one can see that the vector from O to P can be defined as

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$$

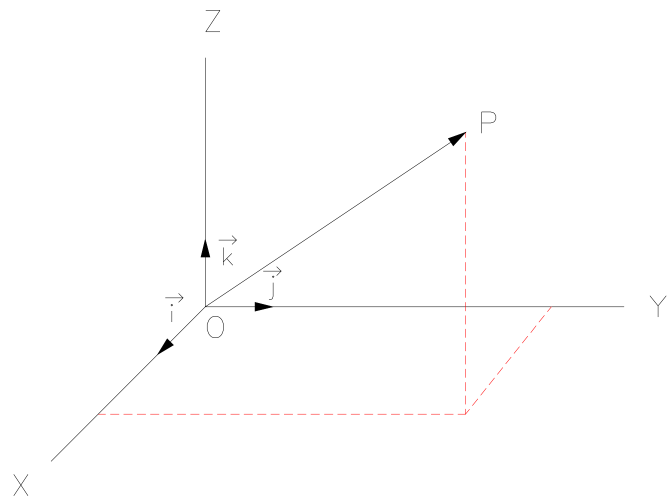

Figure 3. Unit vectors.

and the point P has coordinates $(x, y, z)^T$.

Given a second set of coordinates axes (I, J, K), one can write similar relationships for the same point P. Each coordinate axes has an angular relationship to each of the i, j, k coordinate axes. For example, figure 4 shows the relationship between $\vec{J}$ and $\vec{i}$. The angle between the axes is defined as (xY). Since $\vec{i}$ has similar angles to the other two axes, one can write the unit vector in terms of the direction cosines as:

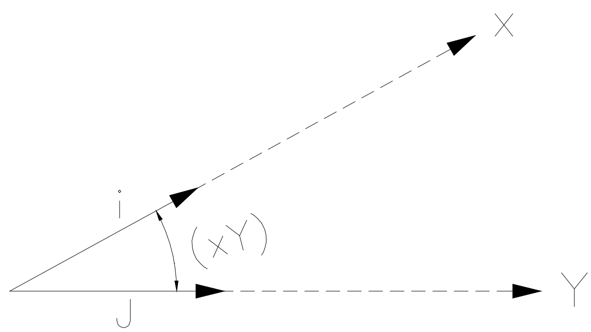

Figure 4. Rotation between Y and x axes.

$$\vec{i} = \begin{pmatrix} \vec{i} \cdot \vec{I} \\ \vec{i} \cdot \vec{J} \\ \vec{i} \cdot \vec{K} \end{pmatrix} = \begin{bmatrix} \cos(xX) \\ \cos(xY) \\ \cos(xZ) \end{bmatrix}$$

Similarly, we have for $\vec{j}$ and $\vec{k}$,

$$\vec{j} = \begin{bmatrix} \cos(yX) \\ \cos(yY) \\ \cos(yZ) \end{bmatrix} \qquad \vec{k} = \begin{bmatrix} \cos(zX) \\ \cos(zY) \\ \cos(zZ) \end{bmatrix}$$

Then, the vector from O to P can be written as

$$\overrightarrow{OP} = x \begin{bmatrix} \cos(xX) \\ \cos(xY) \\ \cos(xZ) \end{bmatrix} + y \begin{bmatrix} \cos(yX) \\ \cos(yY) \\ \cos(yZ) \end{bmatrix} + z \begin{bmatrix} \cos(zX) \\ \cos(zY) \\ \cos(zZ) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(xX) & \cos(yX) & \cos(zX) \\ \cos(xY) & \cos(yY) & \cos(zY) \\ \cos(xZ) & \cos(yZ) & \cos(zZ) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

This can be written more generally as

$$\vec{X} = R\vec{x}$$

To solve these unknowns using only three angles, 6 orthogonal conditions must be applied to the rotation matrix, R. All vectors must have a length of 1 and any combination of the two must be orthogonal [Novak, 1993]. Thus, designating R as three column vectors [R = (r_1 r_2 r_3)], we have

$$r_1^T r_1 = r_2^t r_2 = r_3^T r_3 = 1$$
$$r_1^T r_2 = r_2^T r_3 = r_3^T r_1 = 0$$

Sequential Rotations

Combination	Axes of Rotation
1) Roll (ω) – Pitch (φ) – Yaw (κ)	x – y - z
2) Pitch (φ) – Roll (ω) – Yaw (κ)	y – x - z
3) Heading (H) – Roll (ω) – Pitch (φ)	z – x - y
4) Heading (H) – Pitch (φ) – Roll (ω)	z – y - x
5) Azimuth (α) – Tilt (t) – Swing (s)	z – x - z
6) Azimuth (α) – Elevation (h) – Swing (s)	z – x - z

Table 1. Rotation combinations.

Applying three sequential rotations about three different axes forms the rotation matrix. Doyle [1981] identifies a series of different combinations. These are shown in Table 1 and they all presume a local space coordinate system.

Roll (ω) is a rotation about the x-axis where a positive rotation moves the +y-axis in the direction of the +z-axis. Pitch (φ) is a rotation about the y-axis. When the +z-axis is moved towards the +x-axis then the rotation is positive. A rotation about the z-axis is called yaw (κ) with a positive rotation occurring when the +x-axis is rotated towards the +y-axis. All of these angles have a range from -180° to +180°. Heading (H) is a clockwise rotation about the Z-axis from the +Y-axis to the +X-axis. Azimuth (α) is a clockwise rotation about the Z-axis from the +Y-axis to the principal plane. Tilt (t) is a rotation about the x-axis and is defined as the angle between the camera axis and the nadir or Z-axis. This rotation is positive when the +x-axis is moved towards the +z-axis. Swing is a clockwise angle in the plane of the photograph measured about the z-axis from the +y-axis to the nadir side of the principal line. Heading, azimuth and swing have a range from 0° to 360° while the tilt angle will vary between 0° to 180°. Finally, elevation (h) is a rotation in the vertical plane about the x-axis from the X-Y plane to the camera axis. The rotation is positive when the camera axis is above the X-Y plane.

The combinations (1) and (2) are frequently used in stereoplotters while (3) and (4) are common in navigation. Professor Earl Church developed (5) in his photogrammetric research whereas the ballistic cameras often used the 6th combination.

Derivation of the Gimbal Angles

For a physical interpretation of the rotation matrix written in terms of the directions cosines, we can look at the planar rotations of the axes in sequence. In the first section we saw that the coordinate transformation can be written in the following form:

$$\begin{bmatrix} X_P \\ Y_P \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} U_P \\ V_P \end{bmatrix}$$

In the photogrammetric approach, we rotate the ground coordinates to a photo parallel system. This involves three rotations: ω - primary, φ - secondary, and κ - tertiary. If we look at the ω rotation about the X_1 axis, we should realize that the X-coordinate does not change but the Y and Z coordinates do change (figure 5). Moreover, the new values for Y and Z are not affected by the X-coordinate. Thus, one can write

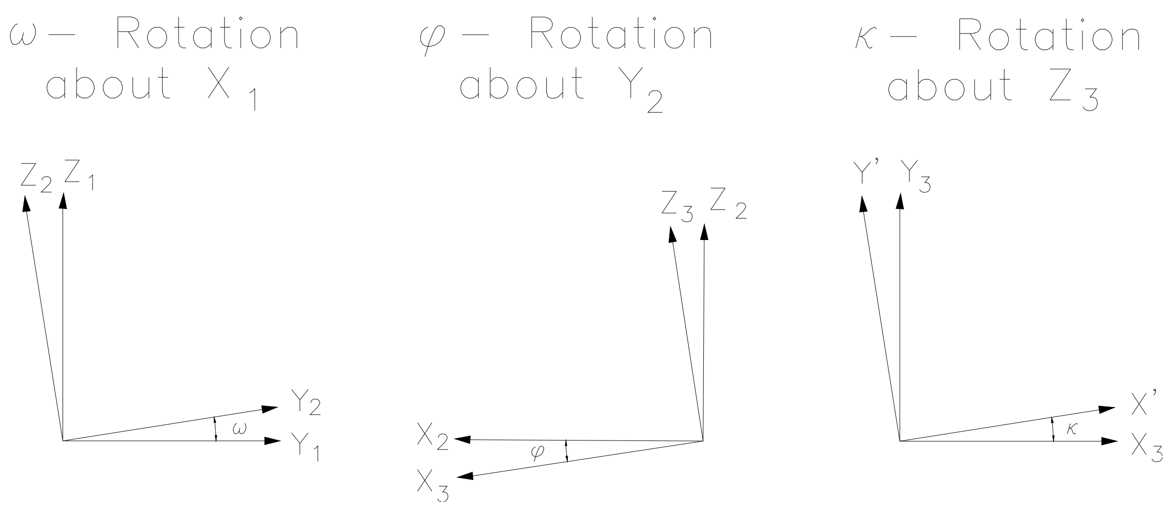

Figure 5. Rotation angles in photogrammetry.

$$
\begin{aligned}
X_2 &= X_1 + Y_1 \cdot 0 + Z_1 \cdot 0 \\
Y_2 &= X_1 \cdot 0 + Y_1 \cdot \cos\omega + Z_1 \cdot \sin\omega \\
Z_2 &= X_1 \cdot 0 + Y_1 \cdot (-\sin\omega) + Z_1 \cdot \cos\omega
\end{aligned}
$$

or in matrix form

$$
\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}
$$

or more concisely,

$$
C_2 = M_\omega C
$$

The next rotation is a φ- rotation about the once rotated Y_2-axis. One can write

$$
\begin{aligned}
X_3 &= X_2 \cdot \cos\varphi + Y_2 \cdot 0 + Z_2 \cdot (-\sin\varphi) \\
Y_3 &= X_2 \cdot 0 + Y_2 + Z_2 \cdot 0 \\
Z_3 &= X_2 \cdot \sin\varphi + Y_2 \cdot 0 + Z_2 \cdot \cos\varphi
\end{aligned}
$$

or in matrix form:

$$
\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}
$$

or more concisely,

$$C_3 = M_\varphi C_2$$

Finally, we have the κ-rotation about the twice-rotated Z_3-axis (see figure 5). This becomes

$$\begin{aligned} X' &= X_3 \cdot \cos\kappa + Y_3 \cdot \sin\kappa + Z_3 \cdot 0 \\ Y' &= X_3 \cdot (-\sin\kappa) + Y_3 \cdot \cos\kappa + Z_3 \cdot 0 \\ Z' &= X_3 \cdot 0 + Y_3 \cdot 0 + Z_3 \end{aligned}$$

which in matrix form is

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix}$$

or more concisely as

$$C' = M_\kappa C_3$$

Thus, the transformation from the survey parallel (X_1, Y_1, Z_1) system is shown as

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = M_G \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = M_\kappa M_\varphi M_\omega \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

Performing the multiplication, the elements of M_G are shown as:

$$M_G = \begin{bmatrix} \cos\varphi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\varphi\cos\kappa & \sin\omega\sin\kappa - \cos\omega\sin\varphi\cos\kappa \\ -\cos\varphi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\varphi\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\varphi\sin\kappa \\ \sin\varphi & -\sin\omega\cos\varphi & \cos\omega\cos\varphi \end{bmatrix}$$

If the rotation matrix is known, then the angles (κ, φ, ω) can be computed as [Doyle, 1981]

$$\tan\omega = \frac{-m_{32}}{m_{33}}$$

$$\sin\phi = m_{31}$$

$$\tan\kappa = \frac{-m_{21}}{m_{11}}$$

If the so-called Church angles (t, s, α) are being used, then the rotation matrix can be derived in a similar fashion. The values for M are:

$$M = \begin{bmatrix} -\cos s\cos\alpha - \cos t\sin\alpha\sin s & \cos s\sin\alpha - \cos t\cos\alpha\sin s & -\sin t\sin s \\ \sin s\cos\alpha - \cos t\sin\alpha\cos s & -\sin s\sin\alpha - \cos t\cos\alpha\cos s & -\sin t\cos s \\ -\sin t\sin\alpha & -\sin t\cos\alpha & \cos t \end{bmatrix}$$

If the rotation matrix is known then the Church angles can be found using the following relationships [Doyle, 1981]:

$$\tan\alpha = \frac{m_{31}}{m_{32}}$$

$$\cos t = m_{33} \qquad \text{or} \qquad \sin t = \sqrt{m_{31}^2 + m_{32}^2} = \sqrt{m_{13}^2 + m_{23}^2}$$

$$\tan s = \frac{m_{13}}{m_{23}}$$

The collinearity concept means that the line form object space to the perspective center is the same as the line from the perspective center to the image point (figure 6). The only difference is a scale factor. Since the comparison is performed in image space, the object space coordinates are rotated into a parallel coordinate system. This relationship can be written as

$$\vec{a} = kM\vec{A}$$

Recall that we wrote two basic equations relating the location of a point in the photo coordinate system and ground nadir position.

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} x - x_o \\ y - y_o \\ -f \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

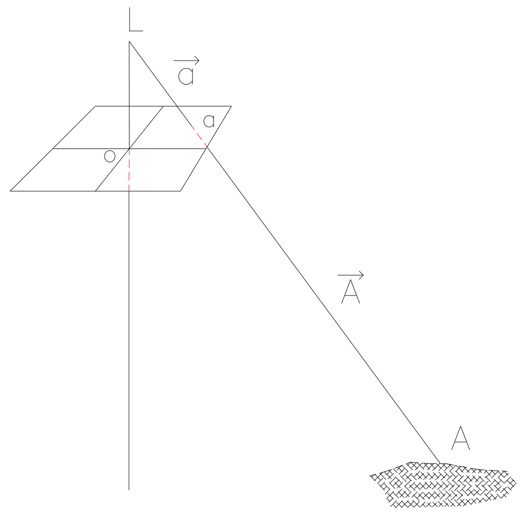

Figure 6. Collinearity condition.

Then,

$$\begin{bmatrix} x - x_o \\ y - y_o \\ -f \end{bmatrix} = k \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

where k is the scale factor. This equation takes the ground coordinates and translates them to the ground nadir position. The rotation matrix (M_G) takes those translated coordinates and rotates them into a system that is parallel to the photograph. Finally, these coordinates are scaled to the photograph. The result is the predicted photo coordinates of the ground points given the exposure station coordinates (X_L, Y_L, Z_L) and the tilt that exists in the photography (κ, φ, ω). If we express this last equation algebraically, then we have

$$\begin{aligned} x - x_o &= k\left[m_{11}(X - X_L) + m_{12}(Y - Y_L) + m_{13}(Z - Z_L)\right] \\ y - y_o &= k\left[m_{21}(X - X_L) + m_{22}(Y - Y_L) + m_{23}(Z - Z_L)\right] \\ -f &= k\left[m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)\right] \end{aligned}$$

To eliminate the unknown scale factor, divide the first two equations by the third. Thus,

$$x - x_o = -f\left[\frac{m_{11}(X - X_L) + m_{12}(Y - Y_L) + m_{13}(Z - Z_L)}{m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)}\right]$$

$$y - y_o = -f\left[\frac{m_{21}(X - X_L) + m_{22}(Y - Y_L) + m_{23}(Z - Z_L)}{m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)}\right]$$

Otto von Gruber first introduced this equation in 1930. This equation must satisfy two conditions [Novak, 1993].

$$m_{11}m_{12} + m_{21}m_{22} + m_{31}m_{32} = 0$$

$$m_{11}^2 + m_{21}^2 + m_{31}^2 = m_{12}^2 + m_{22}^2 + m_{32}^2$$

If we look at the equation for M_G above, lets see if the first condition is met.

$$\begin{aligned} m_{11}m_{12} + m_{21}m_{22} + m_{31}m_{32} &= \cos\varphi\cos\kappa(\cos\omega\sin\kappa + \sin\omega\sin\varphi\cos\kappa) \\ &\quad + (-\cos\varphi\sin\kappa)(\cos\omega\cos\kappa - \sin\xi\text{suin}\varphi\sin\kappa) - \cos\varphi\sin\varphi\sin\omega \\ &= \cos\varphi\cos\kappa\sin\kappa\cos\omega + \cos\varphi\sin\varphi\cos^2\kappa\sin\omega - \cos\varphi\cos\kappa\sin\kappa\cos\omega \\ &\quad + \cos\varphi\sin\varphi\sin^2\kappa\sin\omega - \cos\varphi\sin\varphi\sin\omega \\ &= \cos\varphi\cos\kappa\sin\kappa\cos\omega + \cos\varphi\sin\varphi\sin\omega\left(\cos^2\kappa + \sin^2\kappa\right) \\ &\quad - \cos\varphi\cos\kappa\sin\kappa\cos\omega - \cos\varphi\sin\varphi\sin\omega \\ &= 0 \end{aligned}$$

Thus, the first condition is met. For the second constraint, lets first look at the left hand side of the equation.

$$\begin{aligned} m_{11}^2 + m_{21}^2 + m_{31}^2 &= \cos^2\varphi\cos^2\kappa + \cos^2\varphi\sin^2\kappa + \sin^2\varphi \\ &= \cos^2\varphi\left(\cos^2\kappa + \sin^2\kappa\right) + \sin^2\varphi = 1 \end{aligned}$$

The right side of the equation becomes

$$\begin{aligned} m_{12}^2 + m_{22}^2 + m_{32}^2 &= \cos^2\omega\sin^2\kappa + 2\sin\varphi\cos\kappa\sin\kappa\cos\omega\sin\omega + \sin^2\varphi\cos^2\kappa\sin^2\omega \\ &\quad + \cos^2\kappa\cos^2\omega - 2\sin\varphi\cos\kappa\sin\kappa\cos\omega\sin\omega + \sin\ \ \varphi\sin^2\kappa\sin^2\omega \\ &\quad + \cos^2\varphi\sin^2\omega \\ &= \sin^2\kappa\cos^2\omega + \sin^2\varphi\cos^2\kappa\sin^2\omega + \cos^2\kappa\cos^2\omega + \sin^2\varphi\sin^2\kappa\sin^2\omega \\ &\quad + \cos^2\varphi\sin^2\omega \\ &= \sin^2\varphi\left[\sin^2\omega\left(\cos^2\kappa + \sin^2\kappa\right)\right] + \cos^2\omega + \cos^2\varphi\sin^2\omega \\ &= \sin^2\left(\sin^2\varphi + \cos^2\varphi\right) + \cos^2\omega = 1 \end{aligned}$$

Thus, both sides of the equation equal are equal to one and to each other. Since $(X - X_L)$, $(Y - Y_L)$ and $(Z - Z_L)$ are proportional to the direction cosines of $\vec{A}$, these equations can also be presented as [Doyle, 1981]:

$$x - x_o = -f\ \frac{m_{11}\cos\alpha + m_{12}\cos\beta + m_{13}\cos\gamma}{m_{31}\cos\alpha + m_{32}\cos\beta + m_{33}\cos\gamma}$$

$$y - y_o = -f\ \frac{m_{21}\cos\alpha + m_{22}\cos\beta + m_{23}\cos\gamma}{m_{31}\cos\alpha + m_{32}\cos\beta + m_{33}\cos\gamma}$$

Here, cos α, cos β and cos γ are the direction cosines of $\vec{A}$

The inverse relationship is

$$\begin{aligned} X - X_L &= (Z - Z_L)\left[\frac{m_{11}(x - x_o) + m_{21}(y - y_o + m_{31}(-f))}{m_{13}(x - x_o) + m_{23}(y - y_o + m_{33}(-f))}\right] \\ Y - Y_L &= (Z - Z_L)\left[\frac{m_{12}(x - x_o) + m_{22}(y - y_o + m_{32}(-f))}{m_{13}(x - x_o) + m_{23}(y - y_o + m_{33}(-f))}\right] \end{aligned}$$

These equations are referred to as the collinearity equations.

It would be interesting to see how these equations stand up to the basic principles learned in basic photogrammetry. Recall that for a truly vertical photograph that the scale at a point can be written using

$$S = \frac{f}{H - h} = \frac{x}{X} = \frac{y}{Y}$$

Here we assumed that the principal point coincided with the indicated principal point and that the X and Y ground coordinates were related to the origin, being at the nadir point with the X-axis coinciding with the line from opposite fiducials in the flight direction.

If we look at the collinearity equations, the rotation matrix for a truly vertical photo would be the identity matrix. Thus,

$$M_{Vert} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, the projective equations become

$$\begin{aligned} (x - x_o) &= k\,(X - X_L) \\ (y - y_o) &= k\,(Y - Y_L) \\ -f &= k\,(Z - Z_L) \end{aligned}$$

If we further assume that the principal point is located at the intersection of opposite fiducials and if we substitute H for Z_L and h for Z, then,

$$\begin{aligned} x &= k\left(X - X_L\right) \\ y &= k\left(Y - Y_L\right) \\ f &= k\left(H - h\right) \end{aligned}$$

Dividing the first two equations by the third ad manipulating the equation yields the identical scale relationships given in basic photogrammetry.

$$\frac{x}{X - X_L} = \frac{f}{H - h} = \frac{Y}{Y - Y_L}$$

LINEARIZATION OF THE COLLINEARITY EQUATION

The linearization of the collinearity equations are given in a number of different textbooks. The developments presented here follow that outlined by Doyle [1981]. For simplicity, lets define the projective equations in the following form.

$$F_1 = \left(x - x_o\right) + f\,\frac{U}{W} = 0$$

$$F_2 = \left(y - y_o\right) + f\,\frac{V}{W} = 0$$

where U and V are the numerators in the projective equations given earlier and W is the denominator. From adjustments, we know that the general form of the condition equations can be written as

$$AV + B\Delta + F = 0$$

The deign matrix (B) is found by taking the partial derivative of the projective equations with respect to the parameters. Thus, it will appear as:

$$B = \left[\begin{array}{ccc|cccccc|ccc} \frac{\partial F_1}{\partial x_o} & \frac{\partial F_1}{\partial y_o} & \frac{\partial F_1}{\partial f} & \frac{\partial F_1}{\partial X_L} & \frac{\partial F_1}{\partial Y_L} & \frac{\partial F_1}{\partial Z_L} & \frac{\partial F_1}{\partial \omega} & \frac{\partial F_1}{\partial \varphi} & \frac{\partial F_1}{\partial \kappa} & \frac{\partial F_1}{\partial X_i} & \frac{\partial F_1}{\partial Y_i} & \frac{\partial F_1}{\partial Z_i} \\ \frac{\partial F_2}{\partial x_o} & \frac{\partial F_2}{\partial y_o} & \frac{\partial F_2}{\partial f} & \frac{\partial F_2}{\partial X_L} & \frac{\partial F_2}{\partial Y_L} & \frac{\partial F_2}{\partial Z_L} & \frac{\partial F_2}{\partial \omega} & \frac{\partial F_2}{\partial \varphi} & \frac{\partial F_2}{\partial \kappa} & \frac{\partial F_2}{\partial X_i} & \frac{\partial F_2}{\partial Y_i} & \frac{\partial F_2}{\partial Z_i} \end{array}\right]$$

The first section contains the partial derivatives with respect to the interior orientation, the second group are the partials with respect to the exterior orientation, and the third group are the partials with respect to the ground coordinates. The partial derivatives of the interior orientation (x_o, y_o, and f only) are very basic.

$$\frac{\partial F_1}{\partial x_o} = -1 \qquad \frac{\partial F_1}{\partial y_o} = 0 \qquad \frac{\partial F_1}{\partial f} = \frac{U}{W}$$

$$\frac{\partial F_2}{\partial x_o} = 0 \qquad \frac{\partial F_2}{\partial y_o} = -1 \qquad \frac{\partial F_2}{\partial f} = \frac{V}{W}$$

For the partial derivatives taken with respect to the exposure station coordinates, we will use the following general differentiation formulas:

$$\frac{\partial F_1}{\partial P} = f\,\frac{W\left(\frac{\partial U}{\partial P}\right) - U\left(\frac{\partial W}{\partial P}\right)}{W^2} = \frac{f}{W}\left(\frac{\partial V}{\partial P} - \frac{U}{W}\frac{\partial W}{\partial P}\right)$$

$$\frac{\partial F_2}{\partial P} = f\,\frac{W\left(\frac{\partial V}{\partial P}\right) - V\left(\frac{\partial W}{\partial P}\right)}{W^2} = \frac{f}{W}\left(\frac{\partial V}{\partial P} - \frac{V}{W}\frac{\partial W}{\partial P}\right)$$

where P are the parameters. For the exposure station coordinates (X_L, Y_L, Z_L), the partial derivatives of the functions U, V, and W become:

$$\frac{\partial U}{\partial X_L} = -m_{11} \qquad \frac{\partial U}{\partial Y_L} = -m_{12} \qquad \frac{\partial U}{\partial Z_L} = -m_{13}$$

$$\frac{\partial V}{\partial X_L} = -m_{21} \qquad \frac{\partial V}{\partial Y_L} = -m_{22} \qquad \frac{\partial V}{\partial Z_L} = -m_{23}$$

$$\frac{\partial W}{\partial X_L} = -m_{31} \qquad \frac{\partial W}{\partial Y_L} = -m_{32} \qquad \frac{\partial W}{\partial Z_L} = -m_{33}$$

Then the partial derivatives of the functions F_1 and F_2 can be shown to be

P

$$\frac{\partial F_2}{\partial \omega} = \frac{f}{W}\left(\frac{\partial V}{\partial \omega} - \frac{V}{W}\frac{\partial W}{\partial \omega}\right)$$

$$\frac{\partial F_2}{\partial \varphi} = \frac{f}{W}\left(\frac{\partial V}{\partial \varphi} - \frac{V}{W}\frac{\partial W}{\partial \varphi}\right)$$

$$\frac{\partial F_2}{\partial \kappa} = \frac{f}{W}\left(\frac{\partial V}{\partial \kappa} - \frac{V}{W}\frac{\partial W}{\partial \kappa}\right)$$

The partial derivatives of the functions F_1 and F_2 with respect to the survey points are shown to be:

$$\frac{\partial F_1}{\partial X_i} = -\frac{\partial F_1}{\partial X_L} = \frac{f}{W}\left(m_{11} - \frac{U}{W} m_{31}\right)$$

$$\frac{\partial F_2}{\partial X_i} = -\frac{\partial F_2}{\partial X_L} = \frac{f}{W}\left(m_{21} - \frac{V}{W} m_{31}\right)$$

$$\frac{\partial F_1}{\partial Y_i} = -\frac{\partial F_1}{\partial Y_L} = \frac{f}{W}\left(m_{12} - \frac{U}{W} m_{32}\right)$$

$$\frac{\partial F_2}{\partial Y_i} = -\frac{\partial F_2}{\partial Y_L} = \frac{f}{W}\left(m_{22} - \frac{V}{W} m_{32}\right)$$

$$\frac{\partial}{\partial} = -\frac{\partial}{\partial} = \frac{}{}\left(\quad - \frac{}{} \quad \right)$$