NUMERICAL RESECTION AND ORIENTATION

Center for Photogrammetric Training
Ferris State University

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RCB

INTRODUCTION

- Case I
  - Compute exterior orientation: κ, φ, ω, X_L, Y_L, Z_L
  - Observe photo coordinates: x_i, y_i
  - Treat survey control as known: X_i, Y_i, Z_i
- Case II
  - Extension of Case I – exterior orientation treated as observed quantities

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INTRODUCTION

- Case III
  - Extension of Case II
  - Observed quantities include photo coordinates, eo, survey coordinates (unknown survey points)
  - Survey control given
  - Find adjusted eo and survey coordinates
- Case IV
  - Extension of Case III – interior orientation observed
  - Adjustment: adjusted eo, io, survey coordinates

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INTRODUCTION

- General mathematical model

$$F = F\left(obs,\ X,\ Y\right) = 0$$

- Taylor's series – linearizes equation

$$\mathrm{F} = \mathrm{F}\big|_{\substack{0\\00}} + \left[\frac{\partial \mathrm{F}}{\partial(\mathrm{X})}\right]\Delta\big|_{\substack{0\\00}} + \left[\frac{\partial \mathrm{F}}{\partial(\mathrm{obs})}\right]\Bigg|_{\substack{0\\00}} \mathrm{V} = 0$$

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INTRODUCTION

- Observation equation:

$$F = f + B\Delta + AV = 0$$

$$AV + B\Delta + f = 0$$

- Where
  - V = residuals on the observations
  - Δ = alteration vector to parameters
  - f = discrepancy vector

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CASE I

- Estimate variance-covariance matrix, Σ_{00}
- Compute adjusted eo parameters and variance-covariance matrix on adjusted parameters, $\sum_{00}^{e}$
- Math model
  - central projective equations

$$F(x) = (x - x_o) - c\frac{\Delta X}{\Delta Z} = 0$$

$$F(y) = (y - y_o) - c\frac{\Delta Y}{\Delta Z} = 0$$

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CASE I

- Observation equations

$$A V + \overset{e}{B} \overset{e}{\Delta} + f = 0$$

- where

$$A_j = \left[\frac{\partial F_j}{\partial (obs_j)} \right] = \begin{bmatrix} \dfrac{\partial F(x_j)}{\partial (x_j, y_j)} \\ \dfrac{\partial F(y_j)}{\partial (x_j, y_j)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$V_j = \begin{bmatrix} v_{x_j} \\ v_{y_j} \end{bmatrix} \qquad f = \begin{bmatrix} F(x_j) \\ F(y_j) \end{bmatrix}_{oo}^{o}$$

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CASE I

- Observation equations

$$A V + \overset{e}{B} \overset{e}{\Delta} + f = 0$$

- where

$$\overset{e}{B}_j = \left[\frac{\partial F_j}{\partial (Parameters)} \right] = \begin{bmatrix} \dfrac{\partial F(x_j)}{\partial (\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \\ \dfrac{\partial F(y_j)}{\partial (\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \end{bmatrix}$$

$$\Delta = [\delta\kappa, \delta\varphi, \delta\omega, \delta X_L, \delta Y_L, \delta Z_L]$$

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CASE I

- General form of observation equation

$$V + \overset{e}{B}\overset{e}{\Delta} + f = 0$$

- Function to be minimized

$$F = V^T W V - 2\lambda\left(V + \overset{e}{B}\overset{w}{\Delta} + f\right)$$

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CASE I

- Differentiate the function

$$\frac{\partial F}{\partial V} = 2WV - 2\lambda = 0$$

$$\frac{\partial F}{\partial \overset{e}{\Delta}} = -2\left(\overset{e}{B}\right)^T \lambda = 0$$

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CASE I

- Collecting observation equation and differentiated function

$$\begin{bmatrix} W & 0 & I \\ 0 & 0 & \left(\overset{e}{B}\right)^{-1} \\ I & \overset{e}{B} & 0 \end{bmatrix} \begin{bmatrix} V \\ \overset{e}{\Delta} \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} = 0$$

CASE I

- Eliminating V and λ and substituting V

$$\lambda = -W \overset{e}{B} \overset{e}{\Delta} - Wf$$

- Normal equation found by substituting λ

$$\left(\overset{e}{B}\right)^T W \overset{e}{B} \overset{e}{\Delta} + \left(\overset{e}{B}\right)^T Wf = 0$$

$$N \overset{e}{\Delta} + t = 0$$

CASE I

- Corrections to parameters found by

$$\overset{e}{\Delta} = -N^{-1}t$$

- Adjusted parameters found by adding corrections to current estimates

$$\underset{a}{X} = \underset{oo}{X} + \overset{e}{\Delta}$$

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CASE I

- Residuals computed as $V + f = 0$

$$V = -F_o\Big|_a$$

- Unit variance:

$$\sigma_o^2 = \frac{V^T W V}{2n - 6}$$

- Variance-covariance matrix

$$\sum_{oo}^{\overset{e}{X}} = \sigma_o^2 N^{-1}$$

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EXAMPLE

Photo observations:

Point No.	x	y
1	61.982	79.018
2	-73.147	78.240
3	-54.934	65.899
4	-26.046	-29.449
5	-34.893	-71.287
6	-23.980	-31.889
7	-11.783	88.922
8	-85.047	105.836
9	-26.468	-6.082
10	-12.523	79.026
11	27.972	85.027
12	12.094	-69.861
13	-80.458	-70.012

Survey Control Points:

Point No.	X	Y	Z
1	44646.75000	111295.53700	273.86600
2	45527.20300	109932.63000	275.53100
3	45536.70500	110193.01300	275.10100
4	46322.43000	111086.31900	254.99000
5	46797.22300	111261.00100	263.21400
6	46334.26800	111122.89000	254.85000
7	45019.89000	110475.18200	262.84500
8	45328.04500	109650.87600	291.36500
9	46087.13500	110933.34300	255.65500
10	45126.21800	110531.17400	261.97300
11	44815.80000	110910.16300	288.32000
12	46489.27900	111729.17600	266.85200
13	47061.42300	110795.42700	268.63900

Exterior Orientation Elements (Estimated)

X_L	Y_L	Z_L	Kappa	Phi	Omega
45900.0000	111150.0000	2090.0000	2.1500	0.0000	0.0000

$$N = \begin{bmatrix} \underline{905.6894} & 0.0000 & -65.0311 & 353883.7816 & -1942260.9805 & 106967.6390 \\ & \underline{905.6894} & -195.2187 & -116276.2887 & -106967.6390 & 1886194.4125 \\ & & \underline{291.8256} & 0.0000 & 216476.4928 & -535392.6170 \\ & & & \underline{961373269.1300} & -641531976.7400 & -207855784.9100 \\ & & & & \underline{4273950234.9000} & -479382745.1900 \\ & & & & & \underline{4091640942.8000} \end{bmatrix}$$

$$t = \begin{bmatrix} -51743.527 \\ 14980.210 \\ 326.492 \\ -37790387.580 \\ 108535637.450 \\ 26185558.995 \end{bmatrix}$$

Iteration No. 1
Alteration Vector (Delta):

-8.15331
-3.94869
-0.15855
-0.02176
0.01941
0.00958

Iteration No. 2
Alteration Vector (Delta):

0.61417
0.72201
0.70268
-0.00013
0.00012
0.00022

Iteration No. 3
Alteration Vector (Delta):

0.0015
-0.0015
0.00033
0.00000
0.00000
0.00000

Iteration No. 4
Alteration Vector (Delta):

0.00000
0.00000
0.00000
0.00000
0.00000
0.00000

Exterior Orientation Elements (Adjusted)

X_L	Y_L	Z_L	Kappa	Phi	Omega
45892.4624	111146.7719	2090.5445	2.1281	0.0195	0.0098

Residuals on Photo Observations:

Point No.	x	y
1	-0.002	-0.009
2	0.004	0.007
3	-0.002	0.002
4	-0.001	-0.002
5	0.002	-0.004
6	-0.000	-0.000
7	0.006	0.011
8	0.006	0.001
9	-0.011	-0.000
10	-0.007	0.001
11	0.002	0.006
12	-0.001	0.007
13	0.004	-0.006

The A Posteriori Unit Variance is .3471294

The Variance-Covariance Matrix of Adjusted Parameters is:

.0233948622	.0011026685	-.0020985439	-.0000016307	.0000104961	-.0000002099
.0011026685	.0154028192	-.0034834200	-.0000000932	.0000001678	-.0000075937
-.0020985439	-.0034834200	.0025329779	.0000001566	-.0000009114	.0000018958
-.0000016307	-.0000000932	.0000001566	.0000000005	-.0000000007	.0000000000
.0000104961	.0000001678	-.0000009114	-.0000000007	.0000000048	.0000000001
-.0000002099	-.0000075937	.0000018958	.0000000000	.0000000001	.0000000039

CASE II

- Introduce direct observations on parameters
- Introduce new math model to adjustment

$$F(\kappa) = \underset{o}{\kappa} - \underset{a}{\kappa} = 0$$

$$F(\varphi) = \underset{o}{\varphi} - \underset{a}{\varphi} = 0$$

$$F(\omega) = \underset{o}{\omega} - \underset{a}{\omega} = 0$$

$$F(X_L) = \underset{o}{X_L} - \underset{a}{X_L} = 0$$

$$F(Y_L) = \underset{o}{Y_L} - \underset{a}{Y_L} = 0$$

$$F(Z_L) = \underset{o}{Z_L} - \underset{a}{Z_L} = 0$$

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CASE II

- Observations have residuals, therefore, adjusted parameters only estimated initially

$$\underset{o}{\kappa} + v_\kappa = \underset{oo}{\kappa} + \delta_\kappa$$

$$\underset{o}{\varphi} + v_\varphi = \underset{oo}{\varphi} + \delta_\varphi$$

$$\underset{o}{\omega} + v_\omega = \underset{oo}{\omega} + \delta_\omega$$

$$\underset{o}{X_L} + v_{X_L} = \underset{oo}{X_L} + \delta_{X_L}$$

$$\underset{o}{Y_L} + v_{Y_L} = \underset{oo}{Y_L} + \delta_{Y_L}$$

$$\underset{o}{Z_L} + v_{Z_L} = \underset{oo}{Z_L} + \delta_{Z_L}$$

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CASE II

- Rearrange

$$v_\kappa + \left(\underset{o}{\kappa} - \underset{oo}{\kappa}\right) - \delta_\kappa = 0$$

$$v_\varphi + \left(\underset{o}{\varphi} - \underset{oo}{\varphi}\right) - \delta_\varphi = 0$$

$$v_\omega + \left(\underset{o}{\omega} - \underset{oo}{\omega}\right) - \delta_\omega = 0$$

$$v_{X_L} + \left(\underset{o}{X_L} - \underset{oo}{X_L}\right) - \delta_{X_L} = 0$$

$$v_{Y_L} + \left(\underset{o}{Y_L} - \underset{oo}{Y_L}\right) - \delta_{Y_L} = 0$$

$$v_{Z_L} + \left(\underset{o}{Z_L} - \underset{oo}{Z_L}\right) - \delta_{Z_L} = 0$$

- Observation equations

$$\overset{e}{V} - \overset{e}{\Delta} + \overset{e}{f} = 0$$

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CASE II

- Group with observation equations from projective equations

$$V + \overset{e}{B}\overset{e}{\Delta} + f = 0$$

$$\overset{e}{V} - \overset{e}{\Delta} + \overset{e}{f} = 0$$

- or

$$\overline{V} + \overline{B}\overset{e}{\Delta} + \overline{f} = 0$$

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CASE II

- Function to be minimized:

$$F = \overline{V}^T \overline{W}\,\overline{V} - 2\lambda^T \left(\overline{V} + \overline{B} \overset{e}{\Delta} + \overline{f} \right)$$

- Where weight matrix is

$$\overline{W} = \begin{bmatrix} W & 0 \\ 0 & \overset{e}{W} \end{bmatrix}$$

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CASE II

- Normal equations:

$$\left(\overline{B}^T \overline{W}\,\overline{B} \right) \overset{e}{\Delta} + \overline{B}^T \overline{W}\,\overline{f} = 0$$

- In expanded form:

$$\begin{bmatrix} \overset{e}{B}{}^T & -I \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & \overset{e}{W} \end{bmatrix} \begin{bmatrix} \overset{e}{B} \\ -I \end{bmatrix} \overset{e}{\Delta} + \begin{bmatrix} \overset{e}{B}{}^T & -I \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & \overset{e}{W} \end{bmatrix} \begin{bmatrix} f \\ \overset{e}{f} \end{bmatrix} = 0$$

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CASE II

- Performing multiplication

$$\left(\overset{e}{B}^T W \overset{e}{B} + \overset{e}{W}\right)\overset{e}{\Delta} + \left(\overset{e}{B}^T Wf - \overset{e}{W}\overset{e}{f}\right) = 0$$

- Generally shown as

$$\overline{N}\overset{e}{\Delta} + t = 0$$

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CASE II

- Initial estimate of parameters = observed

$$\overset{a}{\underset{oo}{X}} = \overset{a}{\underset{o}{X}}$$

- Discrepancy vector:

$$\overset{e}{f} = \overset{a}{F}\Big|_{oo}^{o} = 0$$

- Solution:

$$\overset{e}{\Delta} = -\overline{N}^{-1} t$$

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CASE II

- Adjusted parameters: $\underset{a}{X} = \underset{oo}{X} + \overset{e}{\Delta}$
- Residuals: $\overline{V} = -F_{\substack{o\\a}}\Big|$
- Unit variance: $\sigma_o^2 = \dfrac{\overline{V}^T \overline{W}\,\overline{V}}{26-6}$
- A posteriori variance-covariance matrix $\sum_{oo}^{\overset{e}{x}} = \sigma_o^2 \overline{N}^{2}$

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CASE III

- Introduce spatial coordinates as observed
- Math model expanded with survey points

$$F\left(X_j\right) = \underset{o}{X_j} - \underset{a}{X_j} = 0$$

$$F\left(Y_j\right) = \underset{o}{Y_j} - \underset{a}{Y_j} = 0$$

$$F\left(Z_j\right) = \underset{o}{Z_j} - \underset{a}{Z_j} = 0$$

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CASE III

- Observation equations become:

$$\overset{e}{V} - \overset{e}{\Delta} + \overset{e}{f} = 0$$

$$\overset{s}{V} - \overset{s}{\Delta} + \overset{s}{f} = 0$$

$$V + \overset{e}{B}\overset{e}{\Delta} + \overset{s}{B}\overset{s}{\Delta} + f = 0$$

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CASE III

- Observational residuals defined as:

$$\overset{e}{V} = \begin{bmatrix} v_\kappa \\ v_\varphi \\ v_\omega \\ v_{X_L} \\ v_{Y_L} \\ v_{Z_L} \end{bmatrix} \qquad \overset{s}{V} = \begin{bmatrix} v_{X_1} \\ v_{Y_1} \\ v_{Z_1} \\ v_{X_2} \\ \vdots \\ v_{Z_n} \end{bmatrix} \qquad V = \begin{bmatrix} v_{x_1} \\ v_{y_1} \\ v_{x_2} \\ \vdots \\ v_{x_n} \\ x_{y_n} \end{bmatrix}$$

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CASE III

- Discrepancy vectors found by evaluating functions using current estimates

$$f = \begin{bmatrix} F(x_j) \\ \\ F(y_j) \end{bmatrix}_{oo}^{o} \qquad \overset{e}{f} = \begin{bmatrix} F(\kappa) \\ F(\varphi) \\ F(\omega) \\ F(X_L) \\ F(Y_L) \\ F(Z_L) \end{bmatrix}_{oo}^{o} \qquad \overset{s}{f} = \begin{bmatrix} F(X_1) \\ F(Y_1) \\ F(Z_J) \\ \vdots \\ F(X_n) \\ F(Y_n) \\ F(Z_n) \end{bmatrix}$$

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CASE III

- Alteration vectors defined as

$$\overset{e}{\Delta} = \begin{bmatrix} \delta\kappa \\ \delta\varphi \\ \delta\omega \\ \delta X_L \\ \delta Y_L \\ \delta Z_L \end{bmatrix} \qquad \overset{S}{\Delta} = \begin{bmatrix} \delta X_1 \\ \delta Y_1 \\ \delta Z_1 \\ \vdots \\ \delta X_n \\ \delta Y_n \\ \delta Z_n \end{bmatrix}$$

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CASE III

- Design matrices

$$\overset{e}{B} = \begin{bmatrix} \frac{\partial F(x_1)}{\partial(\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \\ \frac{\partial F(y_1)}{\partial(\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \\ \vdots \\ \frac{\partial F(x_n)}{\partial(\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \\ \frac{\partial F(y_n)}{\partial(\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \end{bmatrix} \qquad \overset{s}{B} = \begin{bmatrix} \frac{\partial F(x_1)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \\ \frac{\partial F(y_1)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \\ \frac{\partial F(x_2)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \\ \vdots \\ \frac{\partial F(y_j)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \\ \vdots \\ \frac{\partial F(y_n)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \end{bmatrix}$$

CASE III

- Observation equations

$$\begin{bmatrix} V \\ \overset{e}{V} \\ \overset{s}{V} \end{bmatrix} + \begin{bmatrix} \overset{e}{B} & \overset{s}{B} \\ -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \overset{e}{\Delta} \\ \overset{s}{\Delta} \end{bmatrix} + \begin{bmatrix} f \\ \overset{e}{f} \\ \overset{s}{f} \end{bmatrix} = 0$$

- or

$$\overline{V} + \overline{B}\overline{\Delta} + \overline{f} = 0$$

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CASE III

- Function to be minimized

$$F = \overline{V}^T \overline{W}\,\overline{V} - 2\lambda^T \left(\overline{V} + \overline{B}\,\overline{\Delta} + \overline{f} \right)$$

- Leads to normal equations

$$\left(\overline{B}^T \overline{W}\,\overline{B} \right) \overline{\Delta} + \left(\overline{B}^T \overline{W}\,\overline{f} \right) = 0$$

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CASE III

- Expanded form of normal equations:

$$\begin{bmatrix} \overset{e}{B}^T W \overset{e}{B} + \overset{e}{W} & \overset{e}{B}^T W \overset{s}{B} \\ & \overset{s}{B}^T W \overset{s}{B} + \overset{s}{W} \end{bmatrix} \begin{bmatrix} \overset{e}{\Delta} \\ \overset{s}{\Delta} \end{bmatrix} + \begin{bmatrix} \overset{e}{B}^T Wf - \overset{e}{W}\,\overset{e}{f} \\ \overset{s}{B}^T Wf - \overset{s}{W}\,\overset{s}{f} \end{bmatrix} = 0$$

- Or more simply: $\overline{N}\,\overline{\Delta} + \overline{t} = 0$
- Solution: $\overline{\Delta} = -\overline{N}^{-1}\,\overline{t}$

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Case IV

- Sometimes referred to as "calibration case"
- Additional observations: interior orientation elements, i.e., camera constant (calibrated focal length) and principal point coordinates
  - Can include other items like lens distortion etc
- Math model expanded for c, x_0, y_0

$$F(c) = c_o - c_a = 0$$

$$F(x_0) = x_{0_o} - x_{0_a} = 0$$

$$F(y_0) = y_{0_o} - y_{0_a} = 0$$

Case IV

- Observation equations are:

$$\overset{e}{V} - \overset{e}{\Delta} + \overset{e}{f} = 0$$

$$\overset{s}{V} - \overset{s}{\Delta} + \overset{s}{f} = 0$$

$$\overset{i}{V} - \overset{i}{\Delta} + \overset{i}{f} = 0$$

$$V + \overset{e}{B}\overset{e}{\Delta} + \overset{s}{B}\overset{s}{\Delta} + \overset{i}{B}\overset{i}{\Delta} + f = 0$$

- Or collectively

$$\bar{V} + \bar{B}\bar{\Delta} + \bar{f} = 0$$

Case IV

- The additional observational residuals, discrepancy vector, alteration vector, and design matrix to those defined in Case III are:

$$\overset{i}{f} = \begin{bmatrix} F(c) \\ F(x_0) \\ F(y_0) \end{bmatrix} \qquad \overset{i}{\Delta} = \begin{bmatrix} \delta c \\ \delta x_0 \\ \delta y_0 \end{bmatrix} \qquad \overset{i}{B} = \begin{bmatrix} \dfrac{\partial F(x_i)}{\partial (c, x_0, y_0)} \\ \\ \dfrac{\partial F(y_i)}{\partial (c, x_0, y_0)} \end{bmatrix} \qquad \overset{i}{V} = \begin{bmatrix} v_c \\ v_{x_0} \\ v_{y_0} \end{bmatrix}$$

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Case IV

- Normal equations in expanded form

$$\begin{bmatrix} \underline{\overset{e}{B}{}^T W \overset{e}{B} + \overset{e}{W}} & \overset{e}{B}{}^T W \overset{s}{B} & \overset{e}{B}{}^T W \overset{i}{B} \\ & \underline{\overset{s}{B}{}^T W \overset{s}{B} + \overset{s}{W}} & \overset{s}{B}{}^T W \overset{i}{B} \\ & & \underline{\overset{i}{B}{}^T W \overset{i}{B} + \overset{i}{W}} \end{bmatrix} \begin{bmatrix} \overset{e}{\Delta} \\ \overset{s}{\Delta} \\ \overset{i}{\Delta} \end{bmatrix} + \begin{bmatrix} \overset{e}{B}{}^T W f - \overset{e}{W} \overset{e}{f} \\ \overset{s}{B}{}^T W f - \overset{s}{W} \overset{s}{f} \\ \overset{i}{B}{}^T W f - \overset{i}{W} \overset{i}{f} \end{bmatrix} = 0$$

- Solution as presented previously with collected matrices

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