4 NUMERICAL RESECTION AND ORIENTATION

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Introduction

Numerical resection and orientation involves the determination of the coordinates of the exposure station and the orientation of the photograph in space. Merchant [1973] has identified four different cases, in order of increasing complexity.

Case I: Compute the elements of exterior orientation (κ, φ, ω, X_L, Y_L, and Z_L) by observing the photo coordinates (x_i, y_i) and treating the survey control coordinates (X_i, Y_i, and Z_i) as known.

Case II: This is an extension of Case I with the addition that the elements of exterior orientation are also observed quantities. This can easily be visualized by the use of the global positioning system (GPS) on-board the aircraft.

Case III: This approach is an extension of Case II. Here the observations include photo coordinates, exterior orientation, and survey coordinates (to unknown points). The survey control (coordinates to known points) is given. The solution is to find the adjusted exterior orientation parameters and the survey coordinates.

Case IV: Case IV is a further refinement of Case III except that the elements of interior orientation are observed in addition to the photo coordinates, exterior orientation, and survey coordinates. The adjustment will result in adjusted exterior and interior orientation and survey coordinates.

The general notation for the mathematical model is given as

$$F = F(\text{obs}, X, Y) = 0$$

where: obs = the observed quantities, and
X, Y = the parameters for the condition function.

A Taylor's Series evaluation is done to linearize the equation and this is shown as

$$F = F\big|_{00}^{0} + \left[\frac{\partial F}{\partial (X)}\right]\Delta\big|_{00}^{0} + \left[\frac{\partial F}{\partial (\text{obs})}\right]_{\big|_{00}^{0}} V = 0 \qquad (1)$$

The subscript “0” indicates an observed parameter value whereas “00” means the current estimate of the value. This series is evaluated by comparing the observations to the current estimates of what those values need to be. Evaluation of this function results in the observation equation.

$$F = f + B\Delta + AV = 0$$

or in a more general form:

$$AV + B\Delta + f = 0$$

where: V = the residuals on the observations,
Δ = the alteration vector to the parameters,
f = the discrepancy vector found by comparing the mathematical model using the current estimate of the parameters with the observed values.

Case I

Case I is the simplest form of the space resection problem. The observed values are the photo coordinates (x_i, y_i). The elements of interior orientation along with the survey coordinate control are taken as error free. The observational variance-covariance matrix (3_{oo}) is estimated and the exterior orientation elements ((κ, φ, ω, X_L, Y_L, and Z_L) and variance-covariance matrix on the adjusted parameters (3^{e}_{oo}) are computed. The math model employs the central projective equations as the conditional function. It is shown in general form as:

$$F(x) = \begin{bmatrix} F(x) \\ F(y) \end{bmatrix}$$

where the central projective equations are, for x and y:

$$F(x) = (x - x_o) - c\frac{\Delta X}{\Delta Z} = 0$$

$$F(y) = (y - y_o) - c\frac{\Delta Y}{\Delta Z} = 0$$

The observation equations are written as

$$AV + \overset{e}{B}\overset{e}{\Delta} + f = 0$$

where:

$$A_j = \left[\frac{\partial F_j}{\partial\left(obs_j\right)}\right] = \begin{bmatrix} \dfrac{\partial F\left(x_j\right)}{\partial\left(x_j, y_j\right)} \\ \\ \dfrac{\partial F\left(y_j\right)}{\partial\left(x_j, y_j\right)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\overset{e}{B}_j = \left[\frac{\partial F_j}{\partial\left(\text{Parameters}\right)}\right] = \begin{bmatrix} \dfrac{\partial F\left(x_j\right)}{\partial\left(\kappa, \varphi, \omega, X_L, Y_L, Z_L\right)} \\ \\ \dfrac{\partial F\left(y_j\right)}{\partial\left(\kappa, \varphi, \omega, X_L, Y_L, Z_L\right)} \end{bmatrix}$$

$$V_j = \begin{bmatrix} v_{x_j} \\ v_{y_j} \end{bmatrix} \qquad\qquad f = \begin{bmatrix} F\left(x_j\right) \\ F\left(y_j\right) \end{bmatrix}_{\overset{o}{oo}}$$

$$\Delta = \left[\delta\kappa, \delta\varphi, \delta\omega, \delta X_L, \delta Y_L, \delta Z_L\right]$$

Thus, the general form for n points is

$$V + \overset{e}{B}\overset{e}{\Delta} + f = 0 \qquad (2)$$

If the number of photo points is larger than three then a least squares adjustment is performed. The function to be minimized is expressed as

$$F = V^T W V - 2\lambda\left(V + \overset{e}{B}\overset{w}{\Delta} + f\right)$$

where: λ = the Langrangian multiplier (vector of correlates), and
W = the weight matrix for the photo observations, which is defined as

$$W = \underset{OO}{\overset{obs}{\Sigma}}{}^{-1}$$

The weight matrix is usually assumed to be a diagonal matrix derived from the a priori estimates of the observational variance-covariance matrix. This is usually sufficient for a two-axis comparator but the correlation cannot be neglected for polar comparators.

Differentiation of the function yields

$$\frac{\partial F}{\partial V} = 2\,W\,V - 2\lambda = 0 \qquad (3)$$

$$\frac{\partial F}{\partial \overset{e}{\Delta}} = -2\left(\overset{e}{B}\right)^{T} \lambda = 0 \qquad (4)$$

There are (4n + m) unknowns: 2n in V, 2n in λ, and m in $\overset{e}{\Delta}$. Collecting the observation equation and the differentiated function gives

$$\begin{bmatrix} W & 0 & I \\ 0 & 0 & \left(\overset{e}{B}\right)^{-1} \\ I & \overset{e}{B} & 0 \end{bmatrix} \begin{bmatrix} V \\ \overset{e}{\Delta} \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} = 0 \qquad (5)$$

Eliminating V and λ and substituting V from (3) into (2) yields

$$W^{-1}\lambda + \overset{e}{B}\overset{e}{\Delta} + f = 0$$

or

$$\lambda = -W\,\overset{e}{B}\overset{e}{\Delta} - Wf$$

Substituting λ into (4) results in the normal equations

$$\left(\overset{e}{B}\right)^{T} W\,\overset{e}{B}\overset{e}{\Delta} + \left(\overset{e}{B}\right)^{T} Wf = 0$$

or

$$N\,\overset{e}{\Delta} + t = 0$$

where: N = the normal coefficient matrix and
t = the constant vector

The adjusted parameters become

$$\overset{e}{\Delta} = -N^{-1}t$$

The adjusted parameters are found by adding the corrections to those parameters:

$$X_{a} = X_{oo} + \overset{e}{\Delta}$$

In the least squares adjustment, the process is iterated until the alteration vector reaches some predefined value. The process of updating the parameter values before undergoing another adjustment is commonly referred to as the “Newton-Raphson” method.

The residuals are computed as follows:

$$V + \hat{B}\hat{\Delta} + f = 0$$

Therefore

$$V + f = 0$$

$$V = -F_o\Big|_a$$

The unit variance is expressed as

$$\sigma_o^2 = \frac{V^T W V}{2n - 6}$$

with the variance-covariance matrix relating the adjusted parameters is

$$\sum_{oo}^{\hat{X}} = \sigma_o^2 N^{-1}$$

Example Single Photo Resection – Case 1

Following is an example of a single photo resection and orientation, Case I problem. The following data are entered into the program. Survey control is treated as error free, the photo observations are measured quantities already corrected for atmospheric refraction, lens distortion, and earth curvature. The exterior orientation is estimated. A weight matrix for the photo observations was based on a standard error of 10μm.

Following is an example of the single photo resection and orientation.

SINGLE PHOTO RESECTION AND ORIENTATION - CASE I

Photo Number 1
Photo observations:

Point No.	**x**	**y**
1	61.982	79.018
2	-73.147	78.240
3	-54.934	65.899
4	-26.046	-29.449
5	-34.893	-71.287
6	-23.980	-31.889
7	-11.783	88.922
8	-85.047	105.836
9	-26.468	-6.082
10	-12.523	79.026
11	27.972	85.027
12	12.094	-69.861
13	-80.458	-70.012

Survey Control Points:

Point No.	X	Y	Z
1	44646.75000	111295.53700	273.86600
2	45527.20300	109932.63000	275.53100
3	45536.70500	110193.01300	275.10100
4	46322.43000	111086.31900	254.99000
5	46797.22300	111261.00100	263.21400
6	46334.26800	111122.89000	254.85000
7	45019.89000	110475.18200	262.84500
8	45328.04500	109650.87600	291.36500
9	46087.13500	110933.34300	255.65500
10	45126.21800	110531.17400	261.97300
11	44815.80000	110910.16300	288.32000
12	46489.27900	111729.17600	266.85200
13	47061.42300	110795.42700	268.63900

Exterior Orientation Elements (Estimated)

X_L	Y_L	Z_L	Kappa	Phi	Omega
45900.0000	111150.0000	2090.0000	2.1500	0.0000	0.0000

The initial values for the design matrix (B) and the discrepancy vector (f) are shown as:

$$
B = \begin{bmatrix}
0.0458 & -0.0700 & -0.0372 & 81.1204 & -129.8596 & -132.6350 \\
0.0700 & 0.0458 & -0.0447 & -67.6106 & -183.1953 & 76.7032 \\
0.0459 & -0.0701 & 0.0376 & 81.9612 & -69.1797 & -173.0129 \\
0.0701 & 0.0459 & -0.0452 & 68.2580 & -144.0566 & 138.1935 \\
0.0458 & -0.0701 & 0.0278 & 69.3384 & -73.1100 & -153.8062 \\
0.0701 & 0.0458 & -0.0382 & 50.4256 & -141.0967 & 119.7656 \\
0.0453 & -0.0693 & 0.0128 & -26.3986 & -88.6295 & -128.0349 \\
0.0693 & 0.0453 & 0.0144 & 23.5689 & -133.2941 & 82.2877 \\
0.0456 & -0.0696 & 0.0181 & -67.5381 & -99.4782 & -125.2036 \\
0.0696 & 0.0456 & 0.0370 & 33.1354 & -160.3882 & 87.3077 \\
0.0453 & -0.0693 & 0.0118 & -28.8754 & -88.3078 & -127.5356 \\
0.0693 & 0.0453 & 0.0157 & 21.5686 & -134.0500 & 82.7773 \\
0.0455 & -0.0696 & 0.0038 & 92.0077 & -79.8770 & -129.7678 \\
0.0696 & 0.0455 & -0.0504 & 6.9068 & -171.5356 & 117.1848 \\
0.0463 & -0.0707 & 0.0442 & 109.8029 & -57.8997 & -193.5405 \\
0.0707 & 0.0463 & -0.0610 & 79.5743 & -162.1336 & 174.7222 \\
0.0454 & -0.0694 & 0.0128 & -3.1510 & -85.6027 & -129.9943 \\
0.0694 & 0.0454 & 0.0017 & 23.5140 & -127.5384 & 82.8317 \\
0.0455 & -0.0696 & 0.0043 & 82.0157 & -79.8825 & -129.8732 \\
0.0696 & 0.0455 & -0.0449 & 7.8465 & -161.9332 & 110.9679 \\
0.0462 & -0.0706 & -0.0184 & 87.6316 & -103.1434 & -122.8061 \\
0.0706 & 0.0462 & -0.0486 & -33.1348 & -179.9512 & 94.8692 \\
0.0456 & -0.0698 & -0.0074 & -67.5513 & -78.8336 & -131.5123 \\
0.0698 & 0.0456 & 0.0370 & -13.5210 & -149.0510 & 104.6635 \\
0.0457 & -0.0698 & 0.0427 & -64.9245 & -132.8286 & -142.3670 \\
0.0698 & 0.0457 & 0.0356 & 77.8223 & -168.6172 & 70.5647
\end{bmatrix}
\qquad
f = \begin{bmatrix}
-5.6286 \\ -2.1034 \\ -4.8890 \\ -3.7212 \\ -4.5084 \\ -3.4395 \\ -2.4771 \\ -3.0504 \\ -1.7577 \\ -3.7489 \\ -2.4114 \\ -3.0136 \\ -4.8762 \\ -3.0857 \\ -5.4727 \\ -3.9669 \\ -2.9540 \\ -2.9310 \\ -4.6765 \\ -2.9897 \\ -5.1628 \\ -2.6046 \\ -1.4270 \\ -2.3097 \\ -2.6357 \\ -5.0875
\end{bmatrix}
$$

The initial values for the normal coefficient matrix (N) are:

$$N = \begin{bmatrix} \underline{905.6894} & 0.0000 & -65.0311 & 353883.7816 & -1942260.9805 & 106967.6390 \\ & \underline{905.6894} & -195.2187 & -116276.2887 & -106967.6390 & 1886194.4125 \\ & & \underline{291.8256} & 0.0000 & 216476.4928 & -535392.6170 \\ & & & \underline{961373269.1300} & -641531976.7400 & -207855784.9100 \\ & & & & \underline{4273950234.9000} & -479382745.1900 \\ & & & & & \underline{4091640942.8000} \end{bmatrix}$$

The initial values for the constant vector (t) are compute a $t = B^T W$ and are shown as

$$t = \begin{bmatrix} -51743.527 \\ 14980.210 \\ 326.492 \\ -37790387.580 \\ 108535637.450 \\ 26185558.995 \end{bmatrix}$$

The following data represent the values for the alteration vector for each iteration.

Iteration No. 1
Alteration Vector (Delta):

-8.15331
-3.94869
-0.15855
-0.02176
0.01941
0.00958

Iteration No. 2
Alteration Vector (Delta):

0.61417
0.72201
0.70268
-0.00013
0.00012
0.00022

Iteration No. 3
Alteration Vector (Delta):

0.0015
-0.0015
0.00033
0.00000
0.00000
0.00000

Iteration No. 4
Alteration Vector (Delta):

0.00000
0.00000
0.00000
0.00000
0.00000
0.00000

Exterior Orientation Elements (Adjusted)

X_L	Y_L	Z_L	Kappa	Phi	Omega
45892.4624	111146.7719	2090.5445	2.1281	0.0195	0.0098

Residuals on Photo Observations:

Point No.	x	y
1	-0.002	-0.009
2	0.004	0.007
3	-0.002	0.002
4	-0.001	-0.002
5	0.002	-0.004
6	-0.000	-0.000
7	0.006	0.011
8	0.006	0.001
9	-0.011	-0.000
10	-0.007	0.001
11	0.002	0.006
12	-0.001	0.007
13	0.004	-0.006

The A Posteriori Unit Variance is .3471294

The Variance-Covariance Matrix of Adjusted Parameters is:

.0233948622	.0011026685	-.0020985439	-.0000016307	.0000104961	-.0000002099
.0011026685	.0154028192	-.0034834200	-.0000000932	.0000001678	-.0000075937
-.0020985439	-.0034834200	.0025329779	.0000001566	-.0000009114	.0000018958
-.0000016307	-.0000000932	.0000001566	.0000000005	-.0000000007	.0000000000
.0000104961	.0000001678	-.0000009114	-.0000000007	.0000000048	.0000000001
-.0000002099	-.0000075937	.0000018958	.0000000000	.0000000001	.0000000039

Case II

With Case II, we introduce direct observations on the parameters. A growing example of this situation is the use of airborne GPS where the receiver is used to determine the exposure station of the camera at the instant of exposure. Although this will only provide the exposure station coordinates, integrated systems such as GPS with inertial navigation can yield the rotational elements also. This new resection application adds a new math model to the adjustment. This is, for all of the exterior orientation elements:

$$
\begin{aligned}
F(\kappa) &= \underset{o}{\kappa} - \underset{a}{\kappa} = 0\\
F(\varphi) &= \underset{o}{\varphi} - \underset{a}{\varphi} = 0\\
F(\omega) &= \underset{o}{\omega} - \underset{a}{\omega} = 0\\
F(X_L) &= \underset{o}{X_L} - \underset{a}{X_L} = 0\\
F(Y_L) &= \underset{o}{Y_L} - \underset{a}{Y_L} = 0\\
F(Z_L) &= \underset{o}{Z_L} - \underset{a}{Z_L} = 0
\end{aligned}
$$

Since the observations have residuals, the adjusted parameters can only be estimated initially. Thus,

$$
\begin{aligned}
\underset{o}{\kappa} + v_\kappa &= \underset{oo}{\kappa} + \delta_\kappa\\
\underset{o}{\varphi} + v_\varphi &= \underset{oo}{\varphi} + \delta_\varphi\\
\underset{o}{\omega} + v_\omega &= \underset{oo}{\omega} + \delta_\omega\\
\underset{o}{X_L} + v_{X_L} &= \underset{oo}{X_L} + \delta_{X_L}\\
\underset{o}{Y_L} + v_{Y_L} &= \underset{oo}{Y_L} + \delta_{Y_L}\\
\underset{o}{Z_L} + v_{Z_L} &= \underset{oo}{Z_L} + \delta_{Z_L}
\end{aligned}
$$

Rearranging, we have

$$v_\kappa + \left(\underset{o}{\kappa} - \underset{oo}{\kappa}\right) - \delta_\kappa = 0$$

$$v_\varphi + \left(\underset{o}{\varphi} - \underset{oo}{\varphi}\right) - \delta_\varphi = 0$$

$$v_\omega + \left(\underset{o}{\omega} - \underset{oo}{\omega}\right) - \delta_\omega = 0$$

$$v_{X_L} + \left(\underset{o}{X}_L - \underset{oo}{X}_L\right) - \delta_{X_L} = 0$$

$$v_{Y_L} + \left(\underset{o}{Y}_L - \underset{oo}{Y}_L\right) - \delta_{Y_L} = 0$$

$$v_{Z_L} + \left(\underset{o}{Z}_L - \underset{oo}{Z}_L\right) - \delta_{Z_L} = 0$$

The observation equations are

$$\overset{e}{V} - \overset{e}{\Delta} + \overset{e}{f} = 0$$

Grouped with the observation equations developed for the photo coordinates, we have

$$V + \overset{e}{B}\overset{e}{\Delta} + f = 0$$

$$\overset{e}{V} - \overset{e}{\Delta} + \overset{e}{f} = 0$$

or

$$\overline{V} + \overline{B}\overset{e}{\Delta} + \overline{f} = 0$$

where:

$$\overline{V} = \begin{bmatrix} V \\ \overset{e}{V} \end{bmatrix} \qquad \overline{B} = \begin{bmatrix} \overset{e}{B} \\ -I \end{bmatrix} \qquad \overline{f} = \begin{bmatrix} f \\ \overset{e}{f} \end{bmatrix}$$

The function to be minimized is

$$F = \overline{V}^T \overline{W}\overline{V} - 2\lambda^T\left(\overline{V} + \overline{B}\overset{e}{\Delta} + \overline{f}\right)$$

where the weight matrix is shown to consist of

$$\overline{W} = \begin{bmatrix} W & 0 \\ 0 & \overset{e}{W} \end{bmatrix}$$

The normal equations are then written as

$$\left(\overline{B}^T \overline{W}\overline{B}\right)\overset{e}{\Delta} + \overline{B}^T \overline{W}\overline{f} = 0$$

Which in an expanded form looks like

$$\begin{bmatrix} \overset{e}{B}^T & -I \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & \overset{e}{W} \end{bmatrix} \begin{bmatrix} \overset{e}{B} \\ -I \end{bmatrix} \overset{e}{\Delta} + \begin{bmatrix} \overset{e}{B}^T & -I \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & \overset{e}{W} \end{bmatrix} \begin{bmatrix} f \\ \overset{e}{f} \end{bmatrix} = 0$$

resulting in, after performing the multiplication

$$\left(\overset{e}{B}^T W \overset{e}{B} + \overset{e}{W}\right)\overset{e}{\Delta} + \left(\overset{e}{B}^T Wf - \overset{e}{W}\overset{e}{f}\right) = 0$$

or generally shown as

$$\overline{N}\overset{e}{\Delta} + t = 0$$

On the first cycle in the adjustment the estimates of the parameters are the same as the observed values

$$\overset{a}{X}_{oo} = \overset{a}{X}_{o}$$

Therefore, the discrepancy vector becomes

$$\overset{e}{f} = \overset{a}{F}\Big|_{oo}^{o} = 0$$

Looking at the normal equations, one can see that as the weight matrix for the observed exterior orientation goes to zero then the normal equation reduces to Case I. As before, the solution is expressed as

$$\overset{e}{\Delta} = -\overline{N}^{-1} t$$

The adjusted parameters are computed by adding the discrepancy vector to the current estimate of the parameters.

$$\underset{a}{X} = \underset{oo}{X} + \overset{e}{\Delta}$$

The adjustment is iterated by making these adjusted parameters the current estimates. The cycling continues until the solution reaches some acceptable level.

The residuals are then found by evaluating the function using the observed and final adjusted values.

$$\overline{V} = -F\Big|_{\substack{o\\a}}$$

The unit variance is computed as

$$\sigma_o^2 = \frac{\overline{V}^T \overline{W}\,\overline{V}}{26 - 6}$$

Finally, the a posteriori variance-covariance matrix is found by multiplying the unit variance by N^{-1}.

$$\sum_{oo}^{\substack{e\\x}} = \sigma_o^2 \overline{N}^2$$

Case III

Case III is an extension of Case II in that we now introduce the spatial coordinates as observed quantities thereby constraining the parameters. The math models are:

- For collinearity:

$$F(x) = (x - x_o) - c\frac{\Delta X}{\Delta Z} = 0$$

$$F(y) = (y - y_o) - c\frac{\Delta Y}{\Delta Z} = 0$$

- For the exterior orientation:

$$\begin{aligned}
F(\kappa) &= \underset{o}{\kappa} - \underset{a}{\kappa} = 0 \\
F(\varphi) &= \underset{o}{\varphi} - \underset{a}{\varphi} = 0 \\
F(\omega) &= \underset{o}{\omega} - \underset{a}{\omega} = 0 \\
F(X_L) &= \underset{o}{X_L} - \underset{a}{X_L} = 0 \\
F(Y_L) &= \underset{o}{Y_L} - \underset{a}{Y_L} = 0 \\
F(Z_L) &= \underset{o}{Z_L} - \underset{a}{Z_L} = 0
\end{aligned}$$

- For the ground control:

$$F(X_j) = \underset{o}{X_j} - \underset{a}{X_j} = 0$$
$$F(Y_j) = \underset{o}{Y_j} - \underset{a}{Y_j} = 0$$
$$F(Z_j) = \underset{o}{Z_j} - \underset{a}{Z_j} = 0$$

The observation equations then become

$$\overset{e}{V} - \overset{e}{\Delta} + \overset{e}{f} = 0$$
$$\overset{s}{V} - \overset{s}{\Delta} + \overset{s}{f} = 0$$
$$V + \overset{e}{B}\overset{e}{\Delta} + \overset{s}{B}\overset{s}{\Delta} + f = 0$$

Where the observational residuals on the exterior orientation $\left(\overset{e}{V}\right)$, survey coordinates $\left(\overset{s}{V}\right)$ and photo coordinates (V) are defined as:

$$\overset{e}{V} = \begin{bmatrix} v_\kappa \\ v_\varphi \\ v_\omega \\ v_{X_L} \\ v_{Y_L} \\ v_{Z_L} \end{bmatrix} \qquad \overset{s}{V} = \begin{bmatrix} v_{X_1} \\ v_{Y_1} \\ v_{Z_1} \\ v_{X_2} \\ \vdots \\ v_{Z_n} \end{bmatrix} \qquad V = \begin{bmatrix} v_{x_1} \\ v_{y_1} \\ v_{x_2} \\ \vdots \\ v_{x_n} \\ x_{y_n} \end{bmatrix}$$

The discrepancy vectors $\left(\overset{e}{f}, \overset{s}{f}, f\right)$ are computed by evaluating the functions using the current estimates of the unknown parameters and the original observations.

$$f = \begin{bmatrix} F(x_j) \\ \\ F(y_j) \end{bmatrix}_{\Big|_{oo}^{o}} \qquad \overset{e}{f} = \begin{bmatrix} F(\kappa) \\ F(\varphi) \\ F(\omega) \\ F(X_L) \\ F(Y_L) \\ F(Z_L) \end{bmatrix}_{\Big|_{oo}^{o}} \qquad \overset{s}{f} = \begin{bmatrix} F(X_1) \\ F(Y_1) \\ F(Z_J) \\ \vdots \\ F(X_n) \\ F(Y_n) \\ F(Z_n) \end{bmatrix}$$

The alterations to the current assumed value are shown as:

$$\overset{e}{\Delta} = \begin{bmatrix} \delta\kappa \\ \delta\varphi \\ \delta\omega \\ \delta X_L \\ \delta Y_L \\ \delta Z_L \end{bmatrix} \qquad \overset{s}{\Delta} = \begin{bmatrix} \delta X_1 \\ \delta Y_1 \\ \delta Z_1 \\ \vdots \\ \delta X_n \\ \delta Y_n \\ \delta Z_n \end{bmatrix}$$

The design matrices, $\overset{e}{B}$ and $\overset{s}{B}$, are presented as being

$$\overset{e}{B} = \begin{bmatrix} \dfrac{\partial F(x_1)}{\partial(\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \\ \dfrac{\partial F(y_1)}{\partial(\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \\ \vdots \\ \dfrac{\partial F(x_n)}{\partial(\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \\ \dfrac{\partial F(y_n)}{\partial(\kappa, \varphi, \omega, X_L, Y_L, Z_L)} \end{bmatrix} \qquad \overset{s}{B} = \begin{bmatrix} \dfrac{\partial F(X_1)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \\ \dfrac{\partial F(Y_1)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \\ \dfrac{\partial F(Z_1)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \\ \vdots \\ \dfrac{\partial F(Z_j)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \\ \vdots \\ \dfrac{\partial F(Z_n)}{\partial(X_1, Y_1, Z_1, \cdots, Z_j, \cdots, Z_n)} \end{bmatrix}$$

Collecting the observations

$$\begin{bmatrix} V \\ \overset{e}{V} \\ \overset{s}{V} \end{bmatrix} + \begin{bmatrix} \overset{e}{B} & \overset{s}{B} \\ -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \overset{e}{\Delta} \\ \overset{s}{\Delta} \end{bmatrix} + \begin{bmatrix} f \\ \overset{e}{f} \\ \overset{s}{f} \end{bmatrix} = 0$$

or

$$\overline{V} + \overline{B}\overline{\Delta} + \overline{f} = 0$$

The function to be minimized is

$$F = \overline{V}^T \overline{W}\overline{V} - 2\lambda^T \left(\overline{V} + \overline{B}\overline{\Delta} + \overline{f}\right)$$

This leads to the normal equations

$$\left(\overline{B}^T \overline{W}\overline{B}\right)\overline{\Delta} + \left(\overline{B}^T \overline{W}\overline{f}\right) = 0$$

where the weight matrix is assumed to be free of any correlation and takes the form:

$$W = \begin{bmatrix} W & & \\ & \overset{e}{W} & \\ & & \overset{s}{W} \end{bmatrix}$$

The normal equation in the expanded form is

$$\begin{bmatrix} \overset{e}{B}^T W \overset{e}{B} + \overset{e}{W} & \overset{e}{B}^T W \overset{s}{B} \\ & \overset{s}{B}^T W \overset{s}{B} + \overset{s}{W} \end{bmatrix} \begin{bmatrix} \overset{e}{\Delta} \\ \overset{s}{\Delta} \end{bmatrix} + \begin{bmatrix} \overset{e}{B}^T Wf - \overset{e}{W}\overset{e}{f} \\ \overset{s}{B}^T Wf - \overset{s}{W}\overset{s}{f} \end{bmatrix} = 0$$

or as

$$\overline{N}\overline{\Delta} + \overline{t} = 0$$

The solution is

$$\overline{\Delta} = -\overline{N}^{-1}\overline{t}$$

Then the process is cycled until an acceptable solution is obtained.

References

Merchant, D. C., 1973. "Elements of Photogrammetry, Part II – Computational Photogrammetry, Department of Geodetic Science, Ohio State University, 75p.