COORDINATE TRANSFORMATIONS

The Center for Photogrammetric Training
Ferris State University

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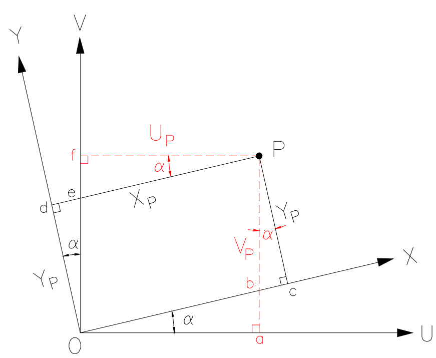
RCB

BASIC PRINCIPLES

- Two coordinate systems: U,V and X,Y



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BASIC PRINCIPLES

- X-coordinate of point P is $X_P = de + eP$
- But, $\tan\alpha = \dfrac{de}{Y_P}$ $\quad \therefore \quad de = Y_P \tan\alpha$

$$\cos\alpha = \frac{U_P}{eP} \qquad \therefore \quad eP = \frac{U_P}{\cos\alpha}$$

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BASIC PRINCIPLES

- Then:

$$X_P = Y_P \tan\alpha + \frac{U_P}{\cos\alpha}$$

$$= \frac{Y_P \sin\alpha}{\cos\alpha} + \frac{U_P}{\cos\alpha}$$

$$X_P \cos\alpha = Y_P \sin\alpha + U_P$$

$$U_P = X_P \cos\alpha - Y_P \sin\alpha$$

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BASIC PRINCIPLES

- For V_P: V_P = ab + Pb
- But, $\tan\alpha = \frac{bc}{Y_P} \Rightarrow bc = Y_P \tan\alpha$

$$\sin\alpha = \frac{ab}{X_P - bc} \Rightarrow ab = (X_P - bc)\sin\alpha$$

- And ab becomes

$$ab = (X_P - Y_P \tan\alpha)\sin\alpha$$

$$= X_P \sin\alpha - Y_P \frac{\sin^2\alpha}{\cos\alpha}$$

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BASIC PRINCIPLES

- Pb can be shown to be

$$\cos\alpha = \frac{Y_P}{Pb} \Rightarrow Pb = \frac{Y_P}{\cos\alpha}$$

- V_P becomes

$$V_P = X_P \sin\alpha - Y_P \frac{\sin^2\alpha}{\cos\alpha} + \frac{Y_P}{\cos\alpha}$$

$$= X_P \sin\alpha + Y_P \left(\frac{1}{\cos\alpha} - \frac{\sin^2\alpha}{\cos\alpha}\right)$$

$$= X_P \sin\alpha + Y_P \cos\alpha$$

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BASIC PRINCIPLES

- Conversion from U,V to X,Y can be developed in a similar fashion

$$X_P = U_P \cos\alpha + V_P \sin\alpha$$

$$Y_P = -U_P \sin\alpha + V_P \cos\alpha$$

- Or in matrix form

$$\begin{bmatrix} X_P \\ Y_P \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} U_P \\ V_P \end{bmatrix}$$

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GENERAL AFFINE TRANSFORMATION

- Normally shown as

$$x' = a_1 x + b_1 y + c_1$$

$$y' = a_2 x + b_2 y + c_2$$

- Unique solution if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

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GENERAL AFFINE TRANSFORMATION

- Used in photogrammetry for:
  - Transform comparator coordinates to photo coordinates and used for correcting film distortion
  - Connecting stereo models
  - Transform model coordinates to survey coordinates

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GENERAL AFFINE TRANSFORMATION

- Property
  - Carry parallel lines into parallel lines
  - Does not have to preserve orthogonality

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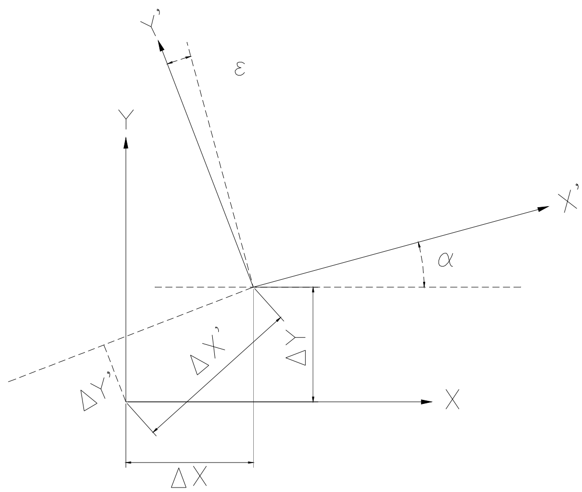
GENERAL AFFINE TRANSFORMATION



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GENERAL AFFINE TRANSFORMATION

- Physical interpretation:

$$x' = C_x(x)\cos\alpha + C_y(y)\sin\alpha + \Delta x'$$

$$y' = -C_x(x)\sin(\alpha+\varepsilon) + C_y(y)\cos(\alpha+\varepsilon) + \Delta y'$$

- 6 parameters: C_x, C_y, α, ε, $\Delta x'$, $\Delta y'$

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GENERAL AFFINE TRANSFORMATION

- Can be related to linear form of the model

$$a_1 = C_x \cos\alpha \qquad a_2 = -C_x \sin(\alpha + \varepsilon)$$
$$b_1 = C_y \sin\alpha \qquad b_2 = C_y \cos(\alpha + \varepsilon)$$
$$c_1 = \Delta x' \qquad c_2 = \Delta y'$$

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GENERAL AFFINE TRANSFORMATION EXAMPLE

- Four fiducial marks (1 - 4) and two image points (a and b) were measured on a comparator. The comparator photo observations and the known values from the camera calibration report are given in the following spreadsheet.

	Photo Coordinates		Known Values	
Point No.	x	y	X	Y
1	-111.734	-114.293	-113.007	-112.997
2	111.734	114.293	113.001	112.989
3	-114.289	111.699	-112.997	113.004
4	114.280	-111.749	112.985	-112.997
a	74.794	12.202		
b	-67.123	53.432		

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GENERAL AFFINE TRANSFORMATION EXAMPLE

6-Parameter Coordinate Transformation Program

Input Values: Note that lower case values represent observed comparator coordinates while the upper case represents the known camera calibration coordinates for the respective fiducial values

$x_1 := -111.734$ $y_1 := -114.293$ $X_1 := -113.007$ $Y_1 := -112.997$

$x_2 := 111.734$ $y_2 := 114.293$ $X_2 := 113.001$ $Y_2 := 112.989$

$x_3 := -114.289$ $y_3 := 111.699$ $X_3 := -112.997$ $Y_3 := 113.004$

$x_4 := 114.280$ $y_4 := -111.749$ $X_4 := 112.985$ $Y_4 := -112.997$

The measured points are:

$x_a := 74.794$ $y_a := 12.202$

$x_b := -67.123$ $y_b := 53.432$

GENERAL AFFINE TRANSFORMATION EXAMPLE

Solution

Forming the B-matrix and f-matrix:

$$B := \begin{pmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \\ x_4 & y_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & y_4 & 1 \end{pmatrix} \qquad f := \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{pmatrix}$$

$$N := \left(B^T B\right)^{-1}$$

GENERAL AFFINE TRANSFORMATION EXAMPLE

The variance-covariance matrix is: $Q_{XX} := N$

$$Q_{XX} = \begin{pmatrix} 19.573E\text{-}006 & -1.603E\text{-}009 & 44.019E\text{-}009 & 0E+000 & 0E+000 & 0E+000 \\ -1.603E\text{-}009 & 19.573E\text{-}006 & 244.661E\text{-}009 & 0E+000 & 0E+000 & 0E+000 \\ 44.019E\text{-}009 & 244.661E\text{-}009 & 250E\text{-}003 & 0E+000 & 0E+000 & 0E+000 \\ 0E+000 & 0E+000 & 0E+000 & 19.573E\text{-}006 & -1.603E\text{-}009 & 44.019E\text{-}009 \\ 0E+000 & 0E+000 & 0E+000 & -1.603E\text{-}009 & 19.573E\text{-}006 & 244.661E\text{-}009 \\ 0E+000 & 0E+000 & 0E+000 & 44.019E\text{-}009 & 244.661E\text{-}009 & 250E\text{-}003 \end{pmatrix}$$

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GENERAL AFFINE TRANSFORMATION EXAMPLE

$t := B^T f$

$$t = \begin{pmatrix} 51079.018 \\ 583.52 \\ -0.018 \\ -578.092 \\ 51078.353 \\ -0.001 \end{pmatrix}$$

The solution vector is: $\Delta := N \cdot t$

$$\Delta = \begin{pmatrix} 0.99977 \\ 0.01134 \\ -0.00211 \\ -0.01140 \\ 0.99977 \\ 0.01222 \end{pmatrix}$$

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GENERAL AFFINE TRANSFORMATION EXAMPLE

The resisuals are $V := B \cdot \Delta - f$

$$V = \begin{pmatrix} 0.001 \\ 0.016 \\ 0.001 \\ 0.016 \\ -0.001 \\ -0.016 \\ -0.001 \\ -0.016 \end{pmatrix}$$

The reference variance for the adjustment is

$$\sigma := \frac{V^T \cdot V}{2} \qquad \sigma = (0.001)$$

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GENERAL AFFINE TRANSFORMATION EXAMPLE

The Transformed coordinates become:

$$X_a := \Delta_1 \cdot x_a + \Delta_2 \cdot y_a + \Delta_3 \qquad X_a = 74.913$$

$$Y_a := \Delta_4 \cdot x_a + \Delta_5 \cdot y_a + \Delta_6 \qquad Y_a = 11.359$$

$$X_b := \Delta_1 \cdot x_b + \Delta_2 \cdot y_b + \Delta_3 \qquad X_b = -66.504$$

$$Y_b := \Delta_4 \cdot x_b + \Delta_5 \cdot y_b + \Delta_6 \qquad Y_b = 54.197$$

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ORTHOGONAL AFFINE TRANSFORMATION

- Impose condition of orthogonality (ε = 0) yielding 5 parameters: C_x, C_y, α, $\Delta x'$, $\Delta y'$

$$x' = C_x\, x \cos\alpha + C_y\, y \sin\alpha + \Delta x'$$

$$y' = -C_x\, x \sin\alpha + C_y\, y \cos\alpha + \Delta y'$$

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ORTHOGONAL AFFINE TRANSFORMATION EXAMPLE

5-Parameter Coordinate Transformation Program

Solution

Forming the B-matrix and f-matrix:

$$B := \begin{pmatrix} x_1 \cdot \cos(\alpha) & y_1 \cdot \sin(\alpha) & -C_x \cdot x_1 \cdot \sin(\alpha) + C_y \cdot y_1 \cdot \cos(\alpha) & 1 & 0 \\ -x_1 \cdot \sin(\alpha) & y_1 \cdot \cos(\alpha) & -C_x \cdot x_1 \cdot \cos(\alpha) - C_y \cdot y_1 \cdot \sin(\alpha) & 0 & 1 \\ x_2 \cdot \cos(\alpha) & y_2 \cdot \sin(\alpha) & -C_x \cdot x_2 \cdot \sin(\alpha) + C_y \cdot y_2 \cdot \cos(\alpha) & 1 & 0 \\ -x_2 \cdot \sin(\alpha) & y_2 \cdot \cos(\alpha) & -C_x \cdot x_2 \cdot \cos(\alpha) - C_y \cdot y_2 \cdot \sin(\alpha) & 0 & 1 \\ x_3 \cdot \cos(\alpha) & y_3 \cdot \sin(\alpha) & -C_x \cdot x_3 \cdot \sin(\alpha) + C_y \cdot y_3 \cdot \cos(\alpha) & 1 & 0 \\ -x_3 \cdot \sin(\alpha) & y_3 \cdot \cos(\alpha) & -C_x \cdot x_3 \cdot \cos(\alpha) - C_y \cdot y_3 \cdot \sin(\alpha) & 0 & 1 \\ x_4 \cdot \cos(\alpha) & y_4 \cdot \sin(\alpha) & -C_x \cdot x_4 \cdot \sin(\alpha) + C_y \cdot y_4 \cdot \cos(\alpha) & 1 & 0 \\ -x_4 \cdot \sin(\alpha) & y_4 \cdot \cos(\alpha) & -C_x \cdot x_4 \cdot \cos(\alpha) - C_y \cdot y_4 \cdot \sin(\alpha) & 0 & 1 \end{pmatrix}$$

$$N := \left(B^T B\right)^{-1}$$

ORTHOGONAL AFFINE TRANSFORMATION EXAMPLE

Introducing some intermediate values, a, c and d

$a := \Delta y \cdot \tan(\alpha) - \Delta x$ $\quad c := \Delta x \cdot \tan(\alpha) + \Delta y$ $\quad d := \cos(\alpha) + \sin(\alpha) \cdot \tan(\alpha)$

$$f := \begin{pmatrix} x_1 - \frac{X_1 - Y_1 \cdot \tan(\alpha) + a}{C_x \cdot d} \\ y_1 - \frac{Y_1 + X_1 \cdot \tan(\alpha) - c}{C_y \cdot d} \\ x_2 - \frac{X_2 - Y_2 \cdot \tan(\alpha) + a}{C_x \cdot d} \\ y_2 - \frac{Y_2 + X_2 \cdot \tan(\alpha) - c}{C_y \cdot d} \\ x_3 - \frac{X_3 - Y_3 \cdot \tan(\alpha) + a}{C_x \cdot d} \\ y_3 - \frac{Y_3 + X_3 \cdot \tan(\alpha) - c}{C_y \cdot d} \\ x_4 - \frac{X_4 - Y_4 \cdot \tan(\alpha) + a}{C_x \cdot d} \\ y_4 - \frac{Y_4 + X_4 \cdot \tan(\alpha) - c}{C_y \cdot d} \end{pmatrix} \qquad f = \begin{pmatrix} 1.273 \\ -1.296 \\ -1.267 \\ 1.304 \\ -1.292 \\ -1.305 \\ 1.295 \\ 1.248 \end{pmatrix}$$

ORTHOGONAL AFFINE TRANSFORMATION EXAMPLE

$t := B^T \cdot f$

$$t = \begin{pmatrix} 11.85 \\ 11.932 \\ -1161.611 \\ 0.009 \\ -0.049 \end{pmatrix}$$

The solution vector is: $\quad \Delta := N \cdot t$

$$\Delta = \begin{pmatrix} 0.00023 \\ 0.00023 \\ -0.01137 \\ 0.00211 \\ -0.01222 \end{pmatrix}$$

Updating the Estimates of the Parameters

$C_x := C_x - \Delta_1 \qquad C_x = 0.9998$

$C_y := C_y - \Delta_2 \qquad C_y = 0.9998$

$\alpha := \alpha - \Delta_3 \qquad \alpha = 0.011368$

$\Delta x := \Delta x - \Delta_4 \qquad \Delta x = -0.0021$

$\Delta y := \Delta y - \Delta_5 \qquad \Delta y = 0.0122$

ORTHOGONAL AFFINE TRANSFORMATION EXAMPLE

- 2nd Iteration (abridged)
  - Solution converged

The variance-covariance matrix is: $Q_{XX} := N$

$$Q_{XX} = \begin{pmatrix} 19.573E\text{-}006 & -65.674E\text{-}015 & -801.884E\text{-}012 & 44.026E\text{-}009 & -498.717E\text{-}012 \\ -65.674E\text{-}015 & 19.573E\text{-}006 & 801.893E\text{-}012 & 2.791E\text{-}009 & 244.647E\text{-}009 \\ -801.884E\text{-}012 & 801.893E\text{-}012 & 9.791E\text{-}006 & 122.1E\text{-}009 & -23.404E\text{-}009 \\ 44.026E\text{-}009 & 2.791E\text{-}009 & 122.1E\text{-}009 & 250E\text{-}003 & -258.227E\text{-}012 \\ -498.717E\text{-}012 & 244.647E\text{-}009 & -23.404E\text{-}009 & -258.227E\text{-}012 & 250E\text{-}003 \end{pmatrix}$$

ORTHOGONAL AFFINE TRANSFORMATION EXAMPLE

The solution vector is: $\Delta := N \cdot t$

$$\Delta = \begin{pmatrix} -0.00007 \\ -0.00006 \\ -0.00000 \\ -0.00000 \\ 0.00000 \end{pmatrix}$$

The parameters are sufficiently small enough to assume that the current estimates are "correct" based on the discripancy values.

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ORTHOGONAL AFFINE TRANSFORMATION EXAMPLE

The resisuals are $V := B \cdot \Delta - f$

$$V = \begin{pmatrix} 0.003 \\ -0.013 \\ -0.004 \\ -0.019 \\ -0.002 \\ 0.02 \\ 0.004 \\ 0.013 \end{pmatrix}$$

The reference variance for the adjustment is

$$\sigma := \frac{V^T \cdot V}{3} \qquad \sigma = (0.000)$$

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ORTHOGONAL AFFINE TRANSFORMATION EXAMPLE

The Transformed coordinates become:

$$X_a := C_x \cdot x_a \cdot \cos(\alpha) + C_y \cdot y_a \cdot \sin(\alpha) + \Delta x \qquad X_a = 74.908$$

$$Y_a := -C_x \cdot x_a \cdot \sin(\alpha) + C_y \cdot y_a \cdot \cos(\alpha) + \Delta y \qquad Y_a = 11.361$$

$$X_b := C_x \cdot x_b \cdot \cos(\alpha) + C_y \cdot y_b \cdot \sin(\alpha) + \Delta x \qquad X_b = -66.498$$

$$Y_b := -C_x \cdot x_b \cdot \sin(\alpha) + C_y \cdot y_b \cdot \cos(\alpha) + \Delta y \qquad Y_b = 54.191$$

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ISOGONAL AFFINE TRANSFORMATION

- Impose additional condition of equal scale ($C = C_x = C_y$) yielding 4 parameters: C, α, $\Delta x'$, $\Delta y'$

$$x' = C\,x\cos\alpha + C\,y\sin\alpha + \Delta x'$$

$$y' = -C\,x\sin\alpha + C\,y\cos\alpha + \Delta y'$$

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ISOGONAL AFFINE TRANSFORMATION

- Recall $C\cos\alpha = a_1 = b_2$

$$-C\sin\alpha = -b_1 = a_2$$

- Then, dropping subscripts, the transformation is:

$$x' = ax + by + c$$

$$y' = -bx + ay + d$$

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ISOGONAL AFFINE TRANSFORMATION

- Back solution

$$x = \frac{a(x' - c) - b(y' - d)}{a^2 + b^2}$$

$$y = \frac{b(x' - c) + a(y' - d)}{a^2 + b^2}$$

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ISOGONAL AFFINE TRANSFORMATION EXAMPLE

4-Parameter Coordinate Transformation Program

Solution

Forming the B-matrix and f-matrix:

$$B := \begin{pmatrix} x_1 & y_1 & 1 & 0 \\ y_1 & -x_1 & 0 & 1 \\ x_2 & y_2 & 1 & 0 \\ y_2 & -x_2 & 0 & 1 \\ x_3 & y_3 & 1 & 0 \\ y_3 & -x_3 & 0 & 1 \\ x_4 & y_4 & 1 & 0 \\ y_4 & -x_4 & 0 & 1 \end{pmatrix} \qquad f := \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{pmatrix}$$

ISOGONAL AFFINE TRANSFORMATION EXAMPLE

$N := \left(B^T B\right)^{-1}$

The variance-covariance matrix is: $Q_{XX} := N$

$$Q_{XX} = \begin{pmatrix} 9.787E\text{-}006 & 0E+000 & 22.02E\text{-}009 & 122.332E\text{-}009 \\ 0E+000 & 9.787E\text{-}006 & 122.332E\text{-}009 & -22.02E\text{-}009 \\ 22.02E\text{-}009 & 122.332E\text{-}009 & 250E\text{-}003 & 0E+000 \\ 122.332E\text{-}009 & -22.02E\text{-}009 & 0E+000 & 250E\text{-}003 \end{pmatrix}$$

$t := B^T f$

$$t = \begin{pmatrix} 102157.371 \\ 1161.611 \\ -0.018 \\ -0.001 \end{pmatrix}$$

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ISOGONAL AFFINE TRANSFORMATION EXAMPLE

The solution vector is: $\Delta := N \cdot t$

$$\Delta = \begin{pmatrix} 0.99977 \\ 0.01137 \\ -0.00211 \\ 0.01222 \end{pmatrix}$$

The resisuals are $V := B \cdot \Delta - f$

$$V = \begin{pmatrix} -0.002 \\ 0.013 \\ 0.004 \\ 0.019 \\ 0.002 \\ -0.020 \\ -0.004 \\ -0.013 \end{pmatrix}$$

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ISOGONAL AFFINE TRANSFORMATION EXAMPLE

The reference variance for the adjustment is

$$\sigma := \frac{V^T \cdot V}{4} \qquad \sigma = (0.0003)$$

The Transformed coordinates become:

$$X_a := \Delta_1 \cdot x_a + \Delta_2 \cdot y_a + \Delta_3 \qquad X_a = 74.913$$

$$Y_a := -\Delta_2 \cdot x_a + \Delta_1 \cdot y_a + \Delta_4 \qquad Y_a = 11.361$$

$$X_b := \Delta_1 \cdot x_b + \Delta_2 \cdot y_b + \Delta_3 \qquad X_b = -66.502$$

$$Y_b := -\Delta_2 \cdot x_b + \Delta_1 \cdot y_b + \Delta_4 \qquad Y_b = 54.195$$

ISOGONAL AFFINE TRANSFORMATION EXAMPLE

- Another approach where the model is not linear

$$B = \begin{bmatrix} \frac{\partial x_1'}{\partial C} & \frac{\partial x_1'}{\partial \alpha} & \frac{\partial x_1'}{\partial \Delta x'} & \frac{\partial x_1'}{\partial \Delta y'} \\ \frac{\partial y_1'}{\partial C} & \frac{\partial y_1'}{\partial \alpha} & \frac{\partial y_1'}{\partial \Delta x'} & \frac{\partial y_1'}{\partial \Delta y'} \\ \frac{\partial x_2'}{\partial C} & \frac{\partial x_2'}{\partial \alpha} & \frac{\partial x_2'}{\partial \Delta x'} & \frac{\partial x_2'}{\partial \Delta y'} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_4'}{\partial C} & \frac{\partial y_4'}{\partial \alpha} & \frac{\partial y_4'}{\partial \Delta x'} & \frac{\partial y_4'}{\partial \Delta y'} \end{bmatrix} \qquad f = \begin{bmatrix} x_1 - \frac{x_1' - y_1' \tan\alpha + \Delta y' \tan\alpha - \Delta x'}{C(\cos\alpha + \sin\alpha \tan\alpha)} \\ y_1 - \frac{y_1' - x_1' \tan\alpha + \Delta x' \tan\alpha - \Delta y'}{C(\cos\alpha + \sin\alpha \tan\alpha)} \\ x_2 - \frac{x_2' - y_2' \tan\alpha + \Delta y' \tan\alpha - \Delta x'}{C(\cos\alpha + \sin\alpha \tan\alpha)} \\ \vdots \\ y_4 - \frac{y_4' - x_4' \tan\alpha + \Delta x' \tan\alpha - \Delta y'}{C(\cos\alpha + \sin\alpha \tan\alpha)} \end{bmatrix}$$

ISOGONAL AFFINE TRANSFORMATION EXAMPLE

Forming the B-matrix and f-matrix:

$$B := \begin{pmatrix} x_1 \cdot \cos(\alpha) + y_1 \cdot \sin(\alpha) & -C \cdot x_1 \cdot \sin(\alpha) + C \cdot y_1 \cdot \cos(\alpha) & 1 & 0 \\ -x_1 \cdot \sin(\alpha) + y_1 \cdot \cos(\alpha) & -C \cdot x_1 \cdot \cos(\alpha) - C \cdot y_1 \cdot \sin(\alpha) & 0 & 1 \\ x_2 \cdot \cos(\alpha) + y_2 \cdot \sin(\alpha) & -C \cdot x_2 \cdot \sin(\alpha) + C \cdot y_2 \cdot \cos(\alpha) & 1 & 0 \\ -x_2 \cdot \sin(\alpha) + y_2 \cdot \cos(\alpha) & -C \cdot x_2 \cdot \cos(\alpha) - C \cdot y_2 \cdot \sin(\alpha) & 0 & 1 \\ x_3 \cdot \cos(\alpha) + y_3 \cdot \sin(\alpha) & -C \cdot x_3 \cdot \sin(\alpha) + C \cdot y_3 \cdot \cos(\alpha) & 1 & 0 \\ -x_3 \cdot \sin(\alpha) + y_3 \cdot \cos(\alpha) & -C \cdot x_3 \cdot \cos(\alpha) - C \cdot y_3 \cdot \sin(\alpha) & 0 & 1 \\ x_4 \cdot \cos(\alpha) + y_4 \cdot \sin(\alpha) & -C \cdot x_4 \cdot \sin(\alpha) + C \cdot y_4 \cdot \cos(\alpha) & 1 & 0 \\ -x_4 \cdot \sin(\alpha) + y_4 \cdot \cos(\alpha) & -C \cdot x_4 \cdot \cos(\alpha) - C \cdot y_4 \cdot \sin(\alpha) & 0 & 1 \end{pmatrix}$$

ISOGONAL AFFINE TRANSFORMATION EXAMPLE

Computing intermediate values

$a := \Delta y \cdot \tan(\alpha) - \Delta x$ $\quad b := \Delta x \cdot \tan(\alpha) - \Delta y$ $\quad d := C \cdot (\cos(\alpha) + \sin(\alpha) \cdot \tan(\alpha))$

$$f := \begin{pmatrix} x_1 - \dfrac{X_1 - Y_1 \cdot \tan(\alpha) + a}{d} \\ y_1 - \dfrac{Y_1 + X_1 \cdot \tan(\alpha) + b}{d} \\ x_2 - \dfrac{X_2 - Y_2 \cdot \tan(\alpha) + a}{d} \\ y_2 - \dfrac{Y_2 + X_2 \cdot \tan(\alpha) + b}{d} \\ x_3 - \dfrac{X_3 - Y_3 \cdot \tan(\alpha) + a}{d} \\ y_3 - \dfrac{Y_3 + X_3 \cdot \tan(\alpha) + b}{d} \\ x_4 - \dfrac{X_4 - Y_4 \cdot \tan(\alpha) + a}{d} \\ y_4 - \dfrac{Y_4 + X_4 \cdot \tan(\alpha) + b}{d} \end{pmatrix} \qquad f = \begin{pmatrix} 1.2730 \\ -1.2960 \\ -1.2670 \\ 1.3040 \\ -1.2920 \\ -1.3050 \\ 1.2950 \\ 1.2480 \end{pmatrix}$$

ISOGONAL AFFINE TRANSFORMATION EXAMPLE

$$N := \left(B^T B\right)^{-1}$$

$$t := B^T f$$

$$t = \begin{pmatrix} 23.781 \\ -1161.611 \\ 0.009 \\ -0.049 \end{pmatrix}$$

The solution vector is: $\Delta := N \cdot t$

$$\Delta = \begin{pmatrix} 0.0002 \\ -0.0114 \\ 0.0021 \\ -0.0122 \end{pmatrix}$$

Updating the current estimates of the parameters:

$C := C - \Delta_1$	$C = 0.9998$
$\alpha := \alpha - \Delta_2$	$\alpha = 0.01137$
$\Delta x := \Delta x - \Delta_3$	$\Delta x = -0.0021$
$\Delta y := \Delta y - \Delta_4$	$\Delta y = 0.0122$

ISOGONAL AFFINE TRANSFORMATION EXAMPLE

- Second iteration (abridged)
  - Converged to solution

$$N := \left(B^T B\right)^{-1}$$

The variance-covariance matrix is: $Q_{XX} := N$

$$Q_{XX} = \begin{pmatrix} 9.787E\text{-}006 & 0E+000 & 23.409E\text{-}009 & 122.074E\text{-}009 \\ 0E+000 & 9.791E\text{-}006 & 122.102E\text{-}009 & -23.414E\text{-}009 \\ 23.409E\text{-}009 & 122.102E\text{-}009 & 250E\text{-}003 & 0E+000 \\ 122.074E\text{-}009 & -23.414E\text{-}009 & 0E+000 & 250E\text{-}003 \end{pmatrix}$$

ISOGONAL AFFINE TRANSFORMATION EXAMPLE

The solution vector is: $\Delta := N \cdot t$

$$\Delta = \begin{pmatrix} -0.0001 \\ -0.0000 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

Updating the current estimates of the parameters:

$C := C - \Delta_1$	$C = 0.9998$
$\alpha := \alpha - \Delta_2$	$\alpha = 0.01137$
$\Delta x := \Delta x - \Delta_3$	$\Delta x = -0.0021$
$\Delta y := \Delta y - \Delta_4$	$\Delta y = 0.0122$

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ISOGONAL AFFINE TRANSFORMATION EXAMPLE

The resisuals are

$$V := B \cdot \Delta - f$$

$$V = \begin{pmatrix} 0.003 \\ -0.013 \\ -0.004 \\ -0.019 \\ -0.003 \\ 0.019 \\ 0.004 \\ 0.013 \end{pmatrix}$$

The reference variance for the adjustment is

$$\sigma := \frac{V^T \cdot V}{4} \qquad \sigma = (0.0003)$$

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ISOGONAL AFFINE TRANSFORMATION EXAMPLE

The Transformed coordinates become:

$$X_a := C \cdot x_a \cdot \cos(\alpha) + C \cdot y_a \cdot \sin(\alpha) + \Delta x \qquad X_a = 74.913$$

$$Y_a := -C \cdot x_a \cdot \sin(\alpha) + C \cdot y_a \cdot \cos(\alpha) + \Delta y \qquad Y_a = 11.361$$

$$X_b := C \cdot x_b \cdot \cos(\alpha) + C \cdot y_b \cdot \sin(\alpha) + \Delta x \qquad X_b = -66.502$$

$$Y_b := -C \cdot x_b \cdot \sin(\alpha) + C \cdot y_b \cdot \cos(\alpha) + \Delta y \qquad Y_b = 54.195$$

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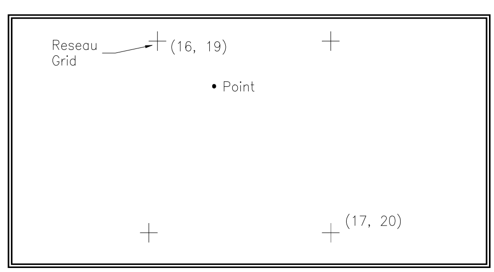
ISOGONAL AFFINE TRANSFORMATION – ANOTHER EXAMPLE

- Measured:
  - x_{UL} = 70.057 mm
  - y_{UL} = -40.014 mm
  - x_{LR} = 80.067 mm
  - y_{LR} = -50.026 mm
  - x_{PT} = 76.0985 mm
  - y_{PT} = -41.9810 mm
- “True”
  - x'_{UL} = 70.107 mm
  - y'_{UL} = -39.843 mm
  - x'_{LR} = 80.133 mm
  - y'_{LR} = -49.820 mm



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ISOGONAL AFFINE TRANSFORMATION

- Differentiating transformation formulas with respect to the parameters:

$$\frac{\partial x'_{UL}}{\partial a} = x_{UL} \qquad \frac{\partial x'_{UL}}{\partial b} = y_{UL}$$

$$\frac{\partial x'_{UL}}{\partial c} = 1 \qquad \frac{\partial x'_{UL}}{\partial d} = 0$$

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ISOGONAL AFFINE TRANSFORMATION

- Design matrix:

$$B = \begin{bmatrix} \frac{\partial x'_{UL}}{\partial (a,b,c,d)} \\ \frac{\partial y'_{UL}}{\partial (a,b,c,d)} \\ \frac{\partial x'_{LR}}{\partial (a,b,c,d)} \\ \frac{\partial y'_{LR}}{\partial (a,b,c,d)} \end{bmatrix} = \begin{bmatrix} x_{UL} & y_{UL} & 1 & 0 \\ y_{UL} & -x_{UL} & 0 & 1 \\ x_{LR} & y_{LR} & 1 & 0 \\ y_{LR} & -x_{LR} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 70.057 & -40.014 & 1 & 0 \\ -40.014 & -70.057 & 0 & 1 \\ 80.067 & -50.026 & 1 & 0 \\ -50.026 & -80.067 & 0 & 1 \end{bmatrix}$$

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ISOGONAL AFFINE TRANSFORMATION

- Discrepancy vector and vector containing the parameters are:

$$f = \begin{bmatrix} x'_{UL} \\ y'_{UL} \\ x'_{LR} \\ y'_{LR} \end{bmatrix} = \begin{bmatrix} 70.107 \\ -39.843 \\ 80.133 \\ -49.820 \end{bmatrix} \qquad \Delta = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

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ISOGONAL AFFINE TRANSFORMATION

- Normal equations: $N = B^TB$

$$N = \begin{bmatrix} 15422.429 & 0 & 150.124 & -90.040 \\ & 15422.429 & -90.040 & -150.124 \\ & & 2 & 0 \\ & & & 2 \end{bmatrix}$$

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ISOGONAL AFFINE TRANSFORMATION

- Constant vector and solution vector are computed as:

$$t = B^T f = \begin{bmatrix} 1514.068 \\ -33.776 \\ 150.240 \\ -89.663 \end{bmatrix} \qquad \Delta = N^{-1} t = \begin{bmatrix} 0.999051 \\ -0.002547 \\ 0.014579 \\ -0.045424 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

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ISOGONAL AFFINE TRANSFORMATION

- Transformed coordinates become:

$$\begin{aligned} x'_{PT} &= a\,x_{PT} + b\,y_{PT} + c \\ &= (0.999051)(76.0985) + (-0.002547)(-41.9810) + 0.014579 \\ &= 76.148 \text{ mm} \end{aligned}$$

$$\begin{aligned} y'_{PT} &= -b\,x_{PT} + a\,y_{PT} + d \\ &= -(-0.002547)(76.0985) + (0.999051)(-41.9810) + (-0.045424) \\ &= -41.793 \end{aligned}$$

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RIGID BODY TRANSFORMATION

- Condition: orthogonality and no scale ($C_x = C_y = 1$)

$$x' = x\cos\alpha + y\sin\alpha + \Delta x'$$

$$y' = -x\sin\alpha + y\cos\alpha + \Delta y'$$

- 3 parameters: α, $\Delta x'$, $\Delta y'$

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RIGID BODY TRANSFORMATION EXAMPLE

3-Parameter Coordinate Transformation Program

Solution

Forming the B-matrix and f-matrix:

$$B := \begin{pmatrix} -x_1\cdot\sin(\alpha) + y_1\cdot\cos(\alpha) & 1 & 0 \\ -x_1\cdot\cos(\alpha) - y_1\cdot\sin(\alpha) & 0 & 1 \\ -x_2\cdot\sin(\alpha) + y_2\cdot\cos(\alpha) & 1 & 0 \\ -x_2\cdot\cos(\alpha) - y_2\cdot\sin(\alpha) & 0 & 1 \\ -x_3\cdot\sin(\alpha) + y_3\cdot\cos(\alpha) & 1 & 0 \\ -x_3\cdot\cos(\alpha) - y_3\cdot\sin(\alpha) & 0 & 1 \\ -x_4\cdot\sin(\alpha) + y_4\cdot\cos(\alpha) & 1 & 0 \\ -x_4\cdot\cos(\alpha) - y_4\cdot\sin(\alpha) & 0 & 1 \end{pmatrix}$$

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RIGID BODY TRANSFORMATION EXAMPLE

$N := (B^T B)^{-1}$

Computing intermediate values for the computation of the f-matrix:

$a := \Delta y \cdot \tan(\alpha) - \Delta x$ $\qquad c := \Delta x \cdot \tan(\alpha) + \Delta y$ $\qquad d := \cos(\alpha) + \sin(\alpha) \cdot \tan(\alpha)$

$$f := \begin{pmatrix} x_1 - \dfrac{X_1 - Y_1 \cdot \tan(\alpha) + a}{d} \\ y_1 - \dfrac{Y_1 + X_1 \cdot \tan(\alpha) - c}{d} \\ x_2 - \dfrac{X_2 - Y_2 \cdot \tan(\alpha) + a}{d} \\ y_2 - \dfrac{Y_2 + X_2 \cdot \tan(\alpha) - c}{d} \\ x_3 - \dfrac{X_3 - Y_3 \cdot \tan(\alpha) + a}{d} \\ y_3 - \dfrac{Y_3 + X_3 \cdot \tan(\alpha) - c}{d} \\ x_4 - \dfrac{X_4 - Y_4 \cdot \tan(\alpha) + a}{d} \\ y_4 - \dfrac{Y_4 + X_4 \cdot \tan(\alpha) - c}{d} \end{pmatrix} \qquad f = \begin{pmatrix} 1.273 \\ -1.296 \\ -1.267 \\ 1.304 \\ -1.292 \\ -1.305 \\ 1.295 \\ 1.248 \end{pmatrix}$$

RIGID BODY TRANSFORMATION EXAMPLE

$t := B^T \cdot f$ $\qquad t = \begin{pmatrix} -1161.611 \\ 0.009 \\ -0.049 \end{pmatrix}$

The solution vector is: $\Delta := N \cdot t$

$$\Delta = \begin{pmatrix} -0.01137 \\ 0.00211 \\ -0.01222 \end{pmatrix}$$

$\alpha := \alpha - \Delta_1$ $\qquad \alpha = 0.01137$

$\Delta x := \Delta x - \Delta_2$ $\qquad \Delta x = -0.0021$

$\Delta y := \Delta y - \Delta_3$ $\qquad \Delta y = 0.0122$

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RIGID BODY TRANSFORMATION EXAMPLE

- 2nd Iteration (abbreviated)

$$N := \left(B^T B\right)^{-1}$$

The variance-covariance matrix is: $Q_{XX} := N$

$$Q_{XX} = \begin{pmatrix} 9.787E\text{-}006 & 122.074E\text{-}009 & -23.409E\text{-}009 \\ 122.074E\text{-}009 & 250E\text{-}003 & -291.994E\text{-}012 \\ -23.409E\text{-}009 & -291.994E\text{-}012 & 250E\text{-}003 \end{pmatrix}$$

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RIGID BODY TRANSFORMATION EXAMPLE

The solution vector is: $\Delta := N \cdot t$

$$\Delta = \begin{pmatrix} -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix}$$

$\alpha := \alpha - \Delta_1$ $\quad \alpha = 0.01137$

$\Delta x := \Delta x - \Delta_2$ $\quad \Delta x = -0.0021$

$\Delta y := \Delta y - \Delta_3$ $\quad \Delta y = 0.0122$

The resisuals are $V := B \cdot \Delta - f$

$$V = \begin{pmatrix} 0.022 \\ 0.006 \\ -0.023 \\ -0.038 \\ 0.016 \\ 0.001 \\ -0.015 \\ 0.032 \end{pmatrix}$$

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RIGID BODY TRANSFORMATION EXAMPLE

The reference variance for the adjustment is

$$\sigma := \frac{V^T \cdot V}{5} \qquad \sigma = (0.001)$$

The Transformed coordinates become:

$$X_a := x_a \cdot \cos(\alpha) + y_a \cdot \sin(\alpha) + \Delta x \qquad X_a = 74.926$$

$$Y_a := -x_a \cdot \sin(\alpha) + y_a \cdot \cos(\alpha) + \Delta y \qquad Y_a = 11.363$$

$$X_b := x_b \cdot \cos(\alpha) + y_b \cdot \sin(\alpha) + \Delta x \qquad X_b = -66.513$$

$$Y_b := -x_b \cdot \sin(\alpha) + y_b \cdot \cos(\alpha) + \Delta y \qquad Y_b = 54.204$$

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POLYNOMIAL TRANSFORMATION

- General form

$$x' = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy + \cdots$$

$$y' = b_0 + b_1 x + bb_2 y + b_3 x^2 + b_4 y^2 + b_5 xy + \cdots$$

- Alternatively,

$$x' = A_0 + A_1 x + A_2 y + A_3 \left(x^2 - y^2\right) + A_4 (2xy) + \cdots$$

$$y' = B_0 - A_2 x + A_1 y + A_4 \left(x^2 - y^2\right) + A_3 (2xy) + \cdots$$

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Bilinear Polynomial Coordinate Transformation Program

Forming the B-matrix and f-matrix:

$$B := \begin{pmatrix} 1 & x_1 & y_1 & x_1 \cdot y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1 \cdot y_1 \\ 1 & x_2 & y_2 & x_2 \cdot y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & x_2 \cdot y_2 \\ 1 & x_3 & y_3 & x_3 \cdot y_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_3 & y_3 & x_3 \cdot y_3 \\ 1 & x_4 & y_4 & x_4 \cdot y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_4 & y_4 & x_4 \cdot y_4 \end{pmatrix} \qquad f := \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{pmatrix}$$

Bilinear Polynomial Coordinate Transformation Program

$$N := \left(B^T B\right)^{-1}$$

The variance-covariance matrix is: $Q_{XX} := N$

$$Q_{XX} = \begin{pmatrix} 0.2500000 & 0.0000000 & 0.0000002 & -0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 \\ 0.0000000 & 0.0000196 & -0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 \\ 0.0000002 & -0.0000000 & 0.0000196 & -0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 \\ -0.0000000 & 0.0000000 & -0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 \\ 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.2500000 & 0.0000000 & 0.0000002 & -0.0000000 \\ 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000196 & -0.0000000 & 0.0000000 \\ 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & 0.0000002 & -0.0000000 & 0.0000196 & -0.0000000 \\ 0.0000000 & 0.0000000 & 0.0000000 & 0.0000000 & -0.0000000 & 0.0000000 & -0.0000000 & 0.0000000 \end{pmatrix}$$

Bilinear Polynomial Coordinate Transformation Program

$t := B^T f$

$$t = \begin{pmatrix} -0.018 \\ 51079.018 \\ 583.52 \\ -455.444 \\ -0.001 \\ -578.092 \\ 51078.353 \\ 340.545 \end{pmatrix}$$

The solution vector is: $\Delta := N \cdot t$

$$\Delta = \begin{pmatrix} -0.0021 \\ 0.9998 \\ 0.0113 \\ -0.0000 \\ 0.0122 \\ -0.0114 \\ 0.9998 \\ -0.0000 \end{pmatrix}$$

Bilinear Polynomial Coordinate Transformation Program

The resisuals are $V := B \cdot \Delta - f$

$$V = \begin{pmatrix} -0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ -0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix}$$

The reference variance for the adjustment is

$$\sigma := \frac{V^T \cdot V}{4} \qquad \sigma = (0.0000)$$

Bilinear Polynomial Coordinate Transformation Program

The Transformed coordinates become:

$$X_a := \Delta_1 + \Delta_2 \cdot x_a + \Delta_3 \cdot y_a + \Delta_4 \cdot x_a \cdot y_a \qquad X_a = 74.913$$

$$Y_a := \Delta_5 + \Delta_6 \cdot x_a + \Delta_7 \cdot y_a + \Delta_8 \cdot x_a \cdot y_a \qquad Y_a = 11.358$$

$$X_b := \Delta_1 + \Delta_2 \cdot x_b + \Delta_3 \cdot y_b + \Delta_4 \cdot x_b \cdot y_b \qquad X_b = -66.503$$

$$Y_b := \Delta_5 + \Delta_6 \cdot x_b + \Delta_7 \cdot y_b + \Delta_8 \cdot x_b \cdot y_b \qquad Y_b = 54.201$$

PROJECTIVE TRANSFORMATION

- Frequently used in photogrammetry
- General form:

$$x' = \frac{a_1 x + a_2 y + a_3}{d_1 x + d_2 y + 1}$$

$$y' = \frac{b_1 x + b_2 y + b_3}{d_1 x + d_2 y + 1}$$

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PROJECTIVE TRANSFORMATION

$$B = \begin{bmatrix} \frac{\partial F(x)_1}{\partial a_1} & \frac{\partial F(x)_1}{\partial a_2} & \cdots & \frac{\partial F(x)_1}{\partial d_2} \\ \frac{\partial F(y)_1}{\partial a_1} & \frac{\partial F(y)_1}{\partial a_2} & \cdots & \frac{\partial F(y)_1}{\partial d_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F(y)_n}{\partial a_1} & \frac{\partial F(y)_n}{\partial a_2} & \cdots & \frac{\partial F(y)_n}{\partial d_2} \end{bmatrix}$$

$$f = \begin{bmatrix} \ell_1 - F(x)_1 \\ \ell_2 - F(y)_1 \\ \vdots \\ \ell_{2n-1} - F(x)_n \\ \ell_{2n} - F(y)_n \end{bmatrix}$$

PROJECTIVE TRANSFORMATION EXAMPLE

Initial estimates of the parameters are given as follows:

$a_1 := 1.0$
$a_2 := 0.0$
$a_3 := 0.0$
$b_1 := 0.0$
$b_2 := 1.0$
$b_3 := 0.0$
$d_1 := 0.0$
$d_2 := 0.0$

The vector of observations

$$L := \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{pmatrix}$$

The denominator of the functional model is:

$$den := \begin{pmatrix} d_1 \cdot x_1 + d_2 \cdot y_1 + 1 \\ d_1 \cdot x_2 + d_2 \cdot y_2 + 1 \\ d_1 \cdot x_3 + d_2 \cdot y_3 + 1 \\ d_1 \cdot x_4 + d_2 \cdot y_4 + 1 \end{pmatrix}$$

PROJECTIVE TRANSFORMATION EXAMPLE

The design matrix, B, is formed as follows:

$$B := \begin{bmatrix}
\frac{x_1}{den_1} & \frac{y_1}{den_1} & \frac{1}{den_1} & 0 & 0 & 0 & -\left[\frac{a_1 \cdot x_1 + a_2 \cdot y_1 + a_3}{(den_1)^2}\right] \cdot x_1 & -\left[\frac{a_1 \cdot x_1 + a_2 \cdot y_1 + a_3}{(den_1)^2}\right] \cdot y_1 \\
0 & 0 & 0 & \frac{x_1}{den_1} & \frac{y_1}{den_1} & \frac{1}{den_1} & -\left[\frac{b_1 \cdot x_1 + b_2 \cdot y_1 + b_3}{(den_1)^2}\right] \cdot x_1 & -\left[\frac{b_1 \cdot x_1 + b_2 \cdot y_1 + b_3}{(den_1)^2}\right] \cdot y_1 \\
\frac{x_2}{den_2} & \frac{y_2}{den_2} & \frac{1}{den_2} & 0 & 0 & 0 & -\left[\frac{a_1 \cdot x_2 + a_2 \cdot y_2 + a_3}{(den_2)^2}\right] \cdot x_2 & -\left[\frac{a_1 \cdot x_2 + a_2 \cdot y_2 + a_3}{(den_2)^2}\right] \cdot y_2 \\
0 & 0 & 0 & \frac{x_2}{den_2} & \frac{y_2}{den_2} & \frac{1}{den_2} & -\left[\frac{b_1 \cdot x_2 + b_2 \cdot y_2 + b_3}{(den_2)^2}\right] \cdot x_2 & -\left[\frac{b_1 \cdot x_2 + b_2 \cdot y_2 + b_3}{(den_2)^2}\right] \cdot y_2 \\
\frac{x_3}{den_3} & \frac{y_3}{den_3} & \frac{1}{den_3} & 0 & 0 & 0 & -\left[\frac{a_1 \cdot x_3 + a_2 \cdot y_3 + a_3}{(den_3)^2}\right] \cdot x_3 & -\left[\frac{a_1 \cdot x_3 + a_2 \cdot y_3 + a_3}{(den_3)^2}\right] \cdot y_3 \\
0 & 0 & 0 & \frac{x_3}{den_3} & \frac{y_3}{den_3} & \frac{1}{den_3} & -\left[\frac{b_1 \cdot x_3 + b_2 \cdot y_3 + b_3}{(den_3)^2}\right] \cdot x_3 & -\left[\frac{b_1 \cdot x_3 + b_2 \cdot y_3 + b_3}{(den_3)^2}\right] \cdot y_3 \\
\frac{x_4}{den_4} & \frac{y_4}{den_4} & \frac{1}{den_4} & 0 & 0 & 0 & -\left[\frac{a_1 \cdot x_4 + a_2 \cdot y_1 + a_3}{(den_4)^2}\right] \cdot x_4 & -\left[\frac{a_1 \cdot x_4 + a_2 \cdot y_4 + a_3}{(den_4)^2}\right] \cdot y_4 \\
0 & 0 & 0 & \frac{x_4}{den_4} & \frac{y_4}{den_4} & \frac{1}{den_4} & -\left[\frac{b_1 \cdot x_4 + b_2 \cdot y_4 + b_3}{(den_4)^2}\right] \cdot x_4 & -\left[\frac{b_1 \cdot x_4 + b_2 \cdot y_4 + b_3}{(den_4)^2}\right] \cdot y_4
\end{bmatrix}$$

PROJECTIVE TRANSFORMATION EXAMPLE

$$F_X := \begin{pmatrix}
\frac{a_1 \cdot x_1 + a_2 \cdot y_1 + a_3}{den_1} \\
\frac{b_1 \cdot x_1 + b_2 \cdot y_1 + b_3}{den_1} \\
\frac{a_1 \cdot x_2 + a_2 \cdot y_2 + a_3}{den_2} \\
\frac{b_1 \cdot x_2 + b_2 \cdot y_2 + b_3}{den_2} \\
\frac{a_1 \cdot x_3 + a_2 \cdot y_3 + a_3}{den_3} \\
\frac{b_1 \cdot x_3 + b_2 \cdot y_3 + b_3}{den_3} \\
\frac{a_1 \cdot x_4 + a_2 \cdot y_4 + a_3}{den_4} \\
\frac{b_1 \cdot x_4 + b_2 \cdot y_4 + b_3}{den_4}
\end{pmatrix}$$

$$f := L - F_X$$

$$f = \begin{pmatrix} -1.273 \\ 1.296 \\ 1.267 \\ -1.304 \\ 1.292 \\ 1.305 \\ -1.295 \\ -1.248 \end{pmatrix}$$

PROJECTIVE TRANSFORMATION EXAMPLE

$N := \left(B^T \cdot B\right)$ $\quad t := B^T \cdot f$ $\quad \Delta := N^{-1} \cdot t$

$$\Delta = \begin{pmatrix} -0.00023 \\ 0.01134 \\ 0.01409 \\ -0.0114 \\ -0.00023 \\ 0.01348 \\ 0 \\ 0 \end{pmatrix}$$

PROJECTIVE TRANSFORMATION EXAMPLE

- Update estimates of parameters and iterate towards a solution

The transformed coordinates are:

$$X_a := \frac{a_1 \cdot x_a + a_2 \cdot y_a + a_3}{d_1 \cdot x_a + d_2 \cdot y_a + 1} \qquad X_a = 74.92187$$

$$Y_a := \frac{b_1 \cdot x_a + b_2 \cdot y_a + b_3}{d_1 \cdot x_a + d_2 \cdot y_a + 1} \qquad Y_a = 11.35877$$

$$X_b := \frac{a_1 \cdot x_b + a_2 \cdot y_b + a_3}{d_1 \cdot x_b + d_2 \cdot y_b + 1} \qquad X_b = -66.49273$$

$$Y_b := \frac{b_1 \cdot x_b + b_2 \cdot y_b + b_3}{d_1 \cdot x_b + d_2 \cdot y_b + 1} \qquad Y_b = 54.20205$$

TRANSFORMATIONS IN THREE DIMENSIONS

- General polynomial approach
  - transformation is not conformal

$$x' = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 yz + a_9 zx + a_{10} xy^2 + a_{11} x^2 y + a_{12} xz^2 + \cdots$$

$$y' = b_0 + b_1 x + b_2 y + b_3 z + b_4 x^2 + b_5 y^2 + b_6 z^2 + b_7 xy + b_8 yz + b_9 zx + b_{10} xy^2 + b_{11} x^2 y + b_{12} xz^2 + \cdots$$

$$z' = c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 y^2 + c_6 z^2 + c_7 xy + c_8 yz + c_9 zx + c_{10} xy^2 + c_{11} x^2 y + c_{12} xz^2 + \cdots$$

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TRANSFORMATIONS IN THREE DIMENSIONS

- Alternative that is conformal in the three planes

$$x' = A_0 + A_1 x + A_2 y + A_3 z + A_5 \left(x^2 - y^2 - z^2\right) + 0 + aA_7 zx + 2A_6 xy + \cdots$$

$$y' = B_0 - A_2 x + A_1 y + A_4 z + A_6 \left(-x^2 + y^2 - z^2\right) + 2A_7 yz + 0 + 2A_5 xy + \cdots$$

$$z' = C_0 + A_3 x - A_4 y + A_1 z + A_7 \left(-x^2 - y^2 + z^2\right) + 2A_6 yz + 2A_5 zx + 0 + \cdots$$

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TRANSFORMATIONS IN THREE DIMENSIONS

- Polynomial projective transformation

$$x' = \frac{a_1 x + a_2 y + a_3 z + a_4}{d_1 x + d_2 y + d_3 z + 1}$$

$$y' = \frac{b_1 x + b_2 y + b_3 z + b_4}{d_1 x + d_2 y + d_3 z + 1}$$

$$z' = \frac{c_1 x + c_2 y + c_3 z + c_4}{d_1 x + d_2 y + d_3 z + 1}$$

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