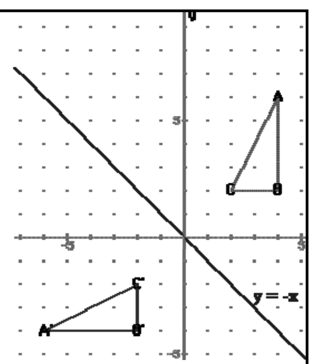
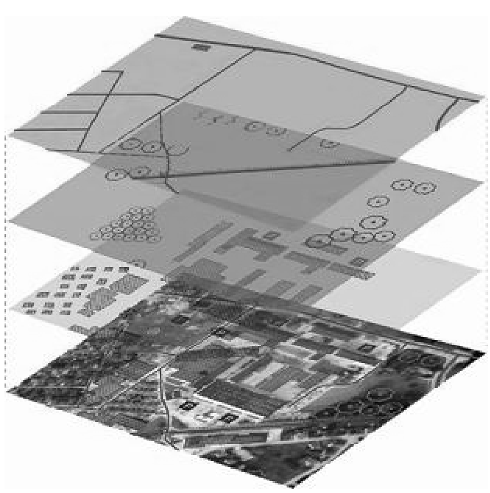
OVERVIEW OF BURSA-WOLF DEVELOPMENT

Surveying Engineering
Ferris State University

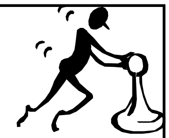
3-D Transformations





Surveying Engineering Department

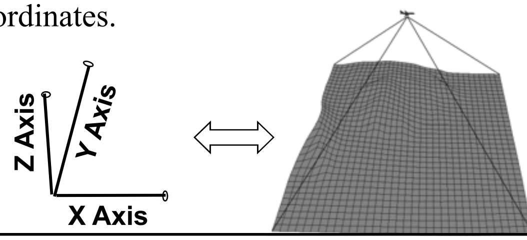
Three-Dimensional Conformal Coordinate Transformation



• Converting from one three-dimensional system to another, while preserving the *true shape*.

• This type of coordinate transformation is essential in analytical **photogrammetry** to transform arbitrary stereo model coordinates to a ground or object space system.

• It is often used in **Geodesy** to convert GPS coordinates in WGS84 to State Plane Coordinates.

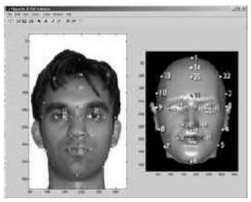


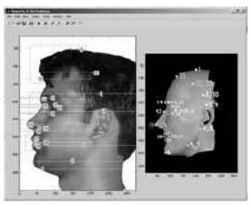
Applications of 3D Conformal Coordinate Transformations

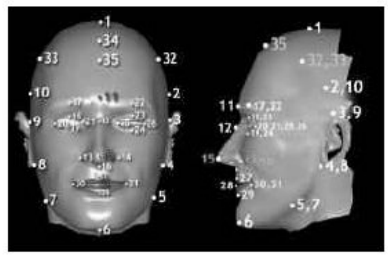
•Mobile mapping systems
- •Relations between different coordinate frames
  - •Sensor frame
  - •Body frame
  - •Mapping frame

Applications of 3D Conformal Coordinate Transformations

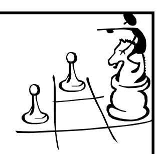
- Homeland security
  - E.G., facial pattern recognition
  - Image processing



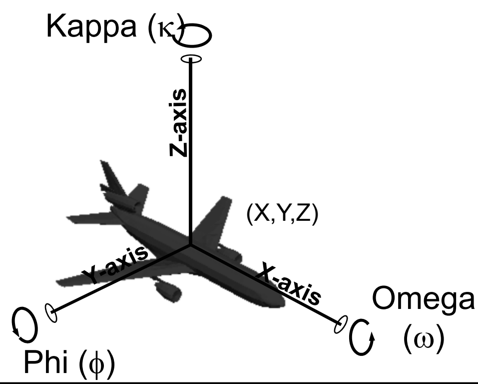




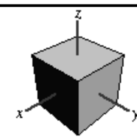
3D Conformal Coordinate Transformation



- Also known as the 7 Parameters transformation since it involves:
- Three rotation angles omega (ω), phi (φ), and kappa (κ);
- Three *translation parameters* (T_X, T_Y, T_Z) and
- a scale factor, S



Rotation angles Omega



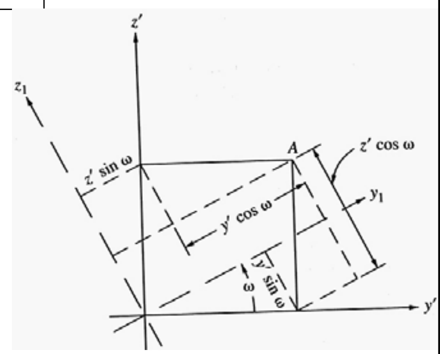
In general form:

$$X_2 = X_1 + Y_1 \cdot 0 + Z_1 \cdot 0$$
$$Y_2 = X_1 \cdot 0 + Y_1 \cdot \cos\omega + Z_1 \cdot \sin\omega$$
$$Z_2 = X_1 \cdot 0 + Y_1 \cdot (-\sin\omega) + Z_1 \cdot \cos\omega$$

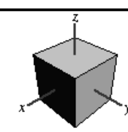
In matrix form:

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

More concisely $C_2 = M_\omega C$



Rotation angles Phi



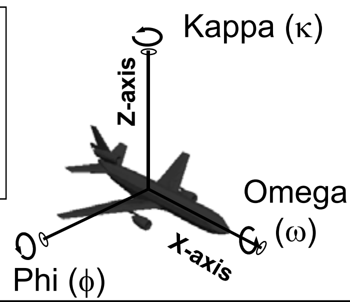
In general form:

$$X_3 = X_2 \cdot \cos\varphi + Y_2 \cdot 0 + Z_2 \cdot (-\sin\varphi)$$
$$Y_3 = X_2 \cdot 0 + Y_2 + Z_2 \cdot 0$$
$$Z_3 = X_2 \cdot \sin\varphi + Y_2 \cdot 0 + Z_2 \cdot \cos\varphi$$

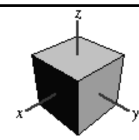
In matrix form:

$$\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

More concisely $C_3 = M_\varphi C_2$



Rotation angles Kappa



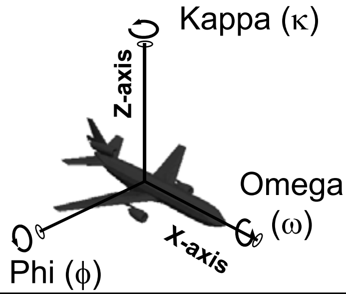
In general form:

$$
\begin{aligned}
X' &= X_3 \cdot \cos\kappa + Y_3 \cdot \sin\kappa + Z_3 \cdot 0 \\
Y' &= X_3 \cdot (-\sin\kappa) + Y_3 \cdot \cos\kappa + Z_3 \cdot 0 \\
Z' &= X_3 \cdot 0 + Y_3 \cdot 0 + Z_3
\end{aligned}
$$

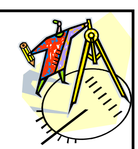
In matrix form:

$$
\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix}
$$

More concisely $C' = M_\kappa C_3$



Combined Rotation Matrix



If we combine all the rotation matrices

$$
\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = M_G \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = M_\kappa M_\varphi M_\omega \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}
$$

$$
M = M_\kappa M_\varphi M_\omega = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}
$$

M_G becomes, after multiplication

$$
M_G = \begin{bmatrix} \cos\varphi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\varphi\cos\kappa & \sin\omega\sin\kappa - \cos\omega\sin\varphi\cos\kappa \\ -\cos\varphi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\varphi\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\varphi\sin\kappa \\ \sin\varphi & -\sin\omega\cos\varphi & \cos\omega\cos\varphi \end{bmatrix}
$$

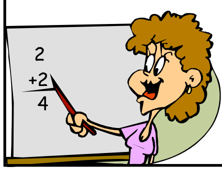
COMPUTING ROTATION ANGLES

- If rotation matrix known, rotation angles can be computed as shown on the right

$$\tan\omega = \frac{-m_{32}}{m_{33}}$$

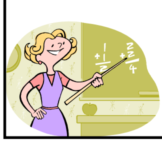
$$\sin\phi = m_{31}$$

$$\tan\kappa = \frac{-m_{21}}{m_{11}}$$

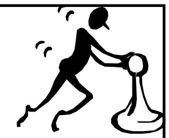


Properties of rotation matrix

- The rotation matrix is an orthogonal matrix, which has the property that its inverse is equal to its transpose, or $R^{-1} = R^{T}$
- This can be used for inverse relationship



Three-Dimensional Conformal Coordinate Transformation

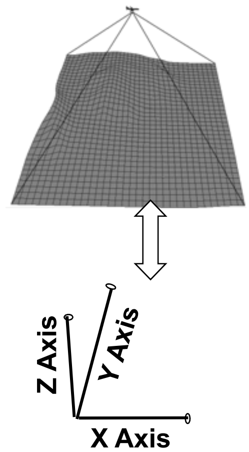


• Finally the 3D Conformal Transformation is derived by multiplying the system by a scale factor *s* and adding the translation factors T_X, T_Y, and T_Z.

$$C' = s \cdot M \cdot C + T$$

• Where:

$$T = \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix} \quad C' = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$



BURSA-WOLF TRANSFORMATION

- Geodesy assumption – rotation angles small
  - $\cos\theta = 1$
  - $\sin\theta = \theta$ (radians)
  - Product of two sines = 0
- Rotation matrix R becomes:

$$R = \begin{bmatrix} 1 & R_\kappa & -R_\varphi \\ -R_\kappa & 1 & R_\omega \\ R_\varphi & -R_\omega & 1 \end{bmatrix}$$

BURSA-WOLF TRANSFORMATION

- 3D similarity transformation

$$\begin{bmatrix} f(x) \\ g(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 & R_\kappa & -R_\varphi \\ -R_\kappa & 0 & R_\omega \\ R_\varphi & -R_\omega & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + s \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

- Observation Equation:

$$\mathbf{V} = \mathbf{B\Delta} - \mathbf{f}$$

BURSA-WOLF TRANSFORMATION

- Coefficient matrix, B:

$$B = \begin{bmatrix} 1 & 0 & 0 & x_1 & 0 & -z_1 & y_1 \\ 0 & 1 & 0 & y_1 & z_1 & 0 & -x_1 \\ 0 & 0 & 1 & z_1 & -y_1 & x_1 & 0 \\ 1 & 0 & 0 & x_2 & 0 & -z_2 & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & z_n & -y_n & x_n & 0 \end{bmatrix}$$

- Vector of parameters, Δ, and discrepancy vector, **f**

$$\mathbf{\Delta} = \begin{bmatrix} T_X & T_Y & T_Z & s & R_\omega & R_\varphi & R_\kappa \end{bmatrix}^T \qquad \mathbf{f} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Three Dimensional Coordinates Transformation

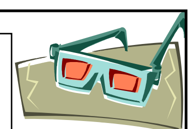
General polynomial approach: transformation is not conformal

$$Xn = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 yz + a_9 zx + a_{10} xy^2 + a_{11} x^2 y + a_{12} xz^2 + \cdots$$

$$Yn = b_0 + b_1 x + b_2 y + b_3 z + b_4 x^2 + b_5 y^2 + b_6 z^2 + b_7 xy + b_8 yz + b_9 zx + b_{10} xy^2 + b_{11} x^2 y + b_{12} xz^2 + \cdots$$

$$Zn = c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 y^2 + c_6 z^2 + c_7 xy + c_8 yz + c_9 zx + c_{10} xy^2 + c_{11} x^2 y + c_{12} xz^2 + \cdots$$

Three Dimensional Coordinates Transformation



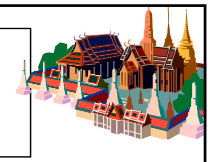
Alternative that is conformal in the three planes

$$Xn = A_0 + A_1 x + A_2 y + A_3 z + A_5 \left(x^2 - y^2 - z^2\right) + 0 + aA_7 zx + 2A_6 xy + \cdots$$

$$Yn = B_0 - A_2 x + A_1 y + A_4 z + A_6 \left(-x^2 + y^2 - z^2\right) + 2A_7 yz + 0 + 2A_5 xy + \cdots$$

$$Zn = C_0 + A_3 x - A_4 y + A_1 z + A_7 \left(-x^2 - y^2 + z^2\right) + 2A_6 yz + 2A_5 zx + 0 + \cdots$$

Three Dimensional Coordinates Transformation



Polynomial projective transformation, 15 parameters

$$Xn = \frac{a_1 x + a_2 y + a_3 z + a_4}{d_1 x + d_2 y + d_3 z + 1}$$

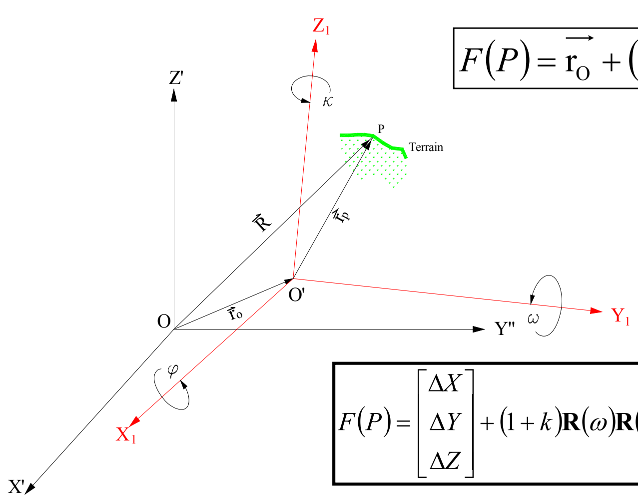
$$Yn = \frac{b_1 x + b_2 y + b_3 z + b_4}{d_1 x + d_2 y + d_3 z + 1}$$

$$Zn = \frac{c_1 x + c_2 y + c_3 z + c_4}{d_1 x + d_2 y + d_3 z + 1}$$

Bursa-Wolf Transformation

From: Krakiwsky and Thomson, 1974

$$F(P) = \overrightarrow{\mathrm{r}_{\mathrm{O}}} + (1+k)\cdot R \cdot \overrightarrow{\mathrm{r}_{\mathrm{P}}} - \vec{\mathrm{R}}$$



$$F(P) = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} + (1+k)\mathbf{R}(\omega)\mathbf{R}(\varphi)\mathbf{R}(\kappa)\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}_P - \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}_P$$

Bursa-Wolf Transformation

- Rotation matrix can be shown as

$$\mathbf{R} = \mathbf{R}(\omega)\mathbf{R}(\varphi)\mathbf{R}(\kappa) = \mathbf{D} = \mathbf{I} + \mathbf{Q}_R$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \kappa & -\omega \\ -\kappa & 0 & \varphi \\ \omega & -\varphi & 0 \end{bmatrix}$$

- Functional model becomes

$$F(P) = \overrightarrow{\mathbf{r_O}} + \overrightarrow{\mathbf{r_P}} + k\overrightarrow{\mathbf{r_P}} + Q_R \overrightarrow{\mathbf{r_P}} - \vec{\mathbf{R}}$$

Bursa-Wolf Transformation

- Linearize – Taylor series

$$\mathbf{AV} + \mathbf{B\Delta} = \mathbf{f} = \mathbf{0}$$

- where

$$\mathbf{f}_i = \overrightarrow{\mathbf{r}_{P_i}} - \overrightarrow{\mathbf{R}}_i$$

$$\mathbf{A} = \frac{\partial F(P)}{\partial \overrightarrow{\mathbf{r}_P}, \overrightarrow{\mathbf{R}}} = [\mathbf{I} \mid -\mathbf{I}]$$

$$\mathbf{B} = \frac{\partial F(P)}{\partial(\Delta X, \Delta Y, \Delta Z, \omega, \varphi, \kappa, k)} = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -Z & Y & X \\ 0 & 1 & 0 & Z & 0 & -X & Y \\ 0 & 0 & 1 & -Y & X & 0 & Z \end{array}\right]$$

Molodensky-Badekas Transformation

- Suited for terrestrial to satellite transformations
- Legacy coordinates: barycentric

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} + \begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} + \begin{bmatrix} 1+\Delta S & \kappa & -\varphi \\ -\kappa & 1+\Delta S & \omega \\ \varphi & -\omega & 1+\Delta S \end{bmatrix} \begin{bmatrix} X_1 - X_B \\ Y_1 - Y_B \\ Z_1 - Z_B \end{bmatrix}$$

GENERALIZED LEAST SQUARES

Surveying Engineering
Ferris State University

INTRODUCTION

- For condition equation of indirect observations – each condition equation contains only 1 observation

$$\mathbf{v} + \mathbf{B}\Delta = \mathbf{f}$$

- In adjustment of observations only, no parameters included in condition

$$\mathbf{A}\mathbf{v} = \mathbf{f}$$

- Generalized least squares – handle combined observations

$$\boxed{\mathbf{A}\mathbf{v} + \mathbf{B}\Delta = \mathbf{f}}$$

REDUNDANCY

- Redundancy, r:

$$\boxed{r = n - n_o}$$

  - n = no. of measurements
  - n_o = min. observations for unique sol.
- To carry unknown parameters in adjustment, need to write additional condition equations for each parameter

$$\boxed{c = r + u}$$

- For u parameters, no. of conditions equations is:

$$0 \le u \le n_o$$
$$r \le c \le n$$

CONDITION EQUATIONS

$$\mathbf{A}(\ell + \mathbf{v}) + \mathbf{B}\Delta = \mathbf{d}$$

- A: c x n coefficient matrix (c ≤ n)
- ℓ: n x 1 observational vector
- v: n x 1 vector of residuals
- B: c x u coefficient matrix (c > u)
- Δ: u x 1 vector of parameters
- d: c x 1 vector of constants

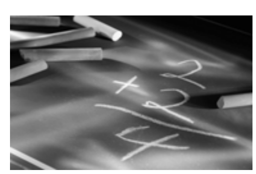
DERIVATION OF LS ESTIMATE OF Δ

Two methods of derivation

1) Condition equations combining observations & parameters transformed into form of indirect observation adjustment
2) Apply minimum criterion directly

Precision expressed as

- Covariance matrix, Σ
- Cofactor matrix, Q
- Weight matrix, W



CONDITION EQUATION FROM COMBINATION

- Vector of c equivalent observations – each linear combination of n original observations

$$\ell_{\mathbf{c}} = \mathbf{A}\ell$$

- The residuals for the corresponding observations

$$\mathbf{v_c} = \mathbf{A}\mathbf{v}$$

- Q_c cofactor of equivalent observations

$$\underset{c,c}{\mathbf{Q}_{\mathbf{c}}} = \underset{c,n}{\mathbf{A}} \cdot \underset{n,n}{\mathbf{Q}} \cdot \underset{n,c}{\mathbf{A}^{\mathrm{T}}}$$

CONDITION EQUATION FROM COMBINATION

- Condition equations

$$\boxed{\begin{aligned} &\ell_c + v_c + B\Delta = d \\ &v_c + B\Delta = d - \ell_c = f \end{aligned}}$$

- Weight matrix of equivalent observations

$$W_e = Q_e^{-1} = \left(AQA^T\right)^{-1}$$

- Normal equation for condition equation

$$N = B^T W_e B = B^T\left(AQA^T\right)^{-1} B$$
$$t = B^T W_e f = B^T\left(AQA^T\right)^{-1} f$$

- Solution: $\boxed{\mathbf{\Delta} = \mathbf{N}^{-1}\mathbf{t}}$

APPLYING MINIMUM CRITERION DIRECTLY

- Minimum criterion:

$$\boldsymbol{\varphi} = \mathbf{v}^{T}\mathbf{W}\mathbf{v} \rightarrow \text{minimum}$$

- Linearized condition equation

$$\boxed{\mathbf{A}\mathbf{v} + \mathbf{B}\boldsymbol{\Delta} = \mathbf{f}}$$

  – where: $\mathbf{A} = \frac{\partial F}{\partial \ell}$; $\mathbf{B} = \frac{\partial F}{\partial x}$; $\mathbf{f} = F(\ell, x_o)$

  – When linear model

$$\mathbf{f} = \mathbf{d} - \mathbf{A}\ell$$

  – When conditions nonlinear

$$\mathbf{f} = F(\ell, x) = 0$$

ℓ is vector of n observations

x is vector of u parameters

APPLYING MINIMUM CRITERION DIRECTLY

- Using vector **k** for c Lagrangian multipliers, minimum criterion is:

$$\boldsymbol{\varphi}' = \mathbf{v}^{T}\mathbf{W}\mathbf{v} - 2\mathbf{k}^{T}(\mathbf{A}\mathbf{v} + \mathbf{B}\boldsymbol{\Delta} - \mathbf{f}) \rightarrow \text{minimum}$$

- To find minimum, differentiate with respect to **v** and Δ and make equal to 0

$$\frac{\partial \boldsymbol{\varphi}'}{\partial \mathbf{v}} = 2\mathbf{v}^{T}\mathbf{W} - 2\mathbf{k}^{T}\mathbf{A} = 0$$

$$\frac{\partial \boldsymbol{\varphi}'}{\partial \boldsymbol{\Delta}} = -2\mathbf{k}^{T}\mathbf{B} = 0$$

APPLYING MINIMUM CRITERION DIRECTLY

- Transpose and rearrange

$$\mathbf{Wv} = \mathbf{A}^T\mathbf{k}$$
$$\mathbf{B}^T\mathbf{k} = \mathbf{0}$$

- Solve for **v** in first equation

$$\mathbf{v} = \mathbf{W}^{-1}\mathbf{A}^T\mathbf{k} = \mathbf{QA}^T\mathbf{k}$$

- Substitute into linearized condition equation

$$\left(\mathbf{AQA}^T\right)\mathbf{k} + \mathbf{B\Delta} = \mathbf{f}$$

- Becomes (in terms of cofactor matrix of equivalent observations)

$$\mathbf{Q}_e\mathbf{k} = \mathbf{f} - \mathbf{B\Delta}$$

APPLYING MINIMUM CRITERION DIRECTLY

- Solve for **k**

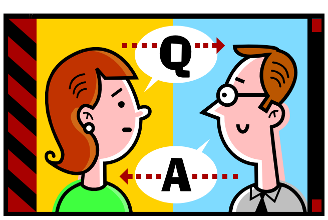
$$\mathbf{k} = \mathbf{Q}_e^{-1}\left(\mathbf{f} - \mathbf{B\Delta}\right) = \mathbf{W}_e\left(\mathbf{f} - \mathbf{B\Delta}\right)$$

- Substitute into equation

$$\mathbf{B}^T\mathbf{k} = 0$$

or

$$\mathbf{B}^T\mathbf{W}_e\left(\mathbf{f} - \mathbf{B\Delta}\right) = 0$$
$$\left(\mathbf{B}^T\mathbf{W}_e\mathbf{B}\right)\mathbf{\Delta} = \mathbf{B}^T\mathbf{W}_e\mathbf{f}$$
$$\mathbf{N\Delta} = \mathbf{t}$$



or

$$\mathbf{N} = \mathbf{B}^T\mathbf{W}_e\mathbf{B} = \mathbf{B}^T\left(\mathbf{AQA}^T\right)^{-1}\mathbf{B}$$
$$\mathbf{t} = \mathbf{B}^T\mathbf{W}_e\mathbf{f} = \mathbf{B}^T\left(\mathbf{AQA}^T\right)^{-1}\mathbf{f}$$

APPLYING MINIMUM CRITERION DIRECTLY

- Solve for Δ
- If condition equations nonlinear, vector estimate of parameters is

$$\hat{\mathbf{x}} = \mathbf{x_o} + \mathbf{\Delta}$$

- Vector of residuals

$$\mathbf{v} = \mathbf{Q}\mathbf{A}^{\mathbf{T}}\mathbf{W_e}\left(\mathbf{f} - \mathbf{B}\mathbf{\Delta}\right)$$

- Vector of adjusted observations

$$\hat{\ell} = \ell + \mathbf{v}$$

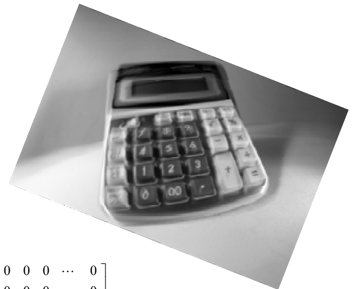
GENERALIZED BURSA-WOLF TRANSFORMATION

- 3D similarity transformation:
  – where

$$\begin{bmatrix} f(x) \\ g(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 & R_\kappa & -R_\varphi \\ -R_\kappa & 0 & R_\omega \\ R_\varphi & -R_\omega & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + s \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$B = \frac{\partial F}{\partial(\Delta X, \Delta Y, \Delta Z, S, \omega, \varphi, \kappa)} = \begin{bmatrix} \frac{\partial f_1}{\partial \Delta X} & \frac{\partial f_1}{\partial \Delta Y} & \frac{\partial f_1}{\partial \Delta Z} & \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial \omega} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial \kappa} \\ \frac{\partial g_1}{\partial \Delta X} & \frac{\partial g_1}{\partial \Delta Y} & \frac{\partial g_1}{\partial \Delta Z} & \frac{\partial g_1}{\partial S} & \frac{\partial g_1}{\partial \omega} & \frac{\partial g_1}{\partial \varphi} & \frac{\partial g_1}{\partial \kappa} \\ \frac{\partial h_1}{\partial \Delta X} & \frac{\partial h_1}{\partial \Delta Y} & \frac{\partial h_1}{\partial \Delta Z} & \frac{\partial h_1}{\partial S} & \frac{\partial h_1}{\partial \omega} & \frac{\partial h_1}{\partial \varphi} & \frac{\partial h_1}{\partial \kappa} \\ \frac{\partial f_2}{\partial \Delta X} & \frac{\partial f_2}{\partial \Delta Y} & \frac{\partial f_2}{\partial \Delta Z} & \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial \omega} & \frac{\partial f_2}{\partial \varphi} & \frac{\partial f_2}{\partial \kappa} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_n}{\partial \Delta X} & \frac{\partial h_n}{\partial \Delta Y} & \frac{\partial h_n}{\partial \Delta Z} & \frac{\partial h_n}{\partial S} & \frac{\partial f_1}{\partial \omega} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial \kappa} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & X_{1_1} & 0 & -Z_{1_1} & Y_{1_1} \\ 0 & 1 & 0 & Y_{1_1} & Z_{1_1} & 0 & -X_{1_1} \\ 0 & 0 & 1 & Z_{1_1} & -Y_{1_1} & X_{1_1} & 0 \\ 1 & 0 & 0 & X_{1_2} & 0 & -Z_{1_2} & Y_{1_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & Z_{1_n} & -Y_{1_n} & X_{1_n} & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f(x_1)}{\partial X'_1} & \frac{\partial f(x_1)}{\partial Y'_1} & \frac{\partial f(x_1)}{\partial Z'_1} & \frac{\partial f(x_1)}{\partial X_{1_1}} & \frac{\partial f(x_1)}{\partial Y_{1_1}} & \frac{\partial f(x_1)}{\partial Z_{1_1}} & \frac{\partial f(x_1)}{\partial X'_2} & \frac{\partial f(x_1)}{\partial Y'_2} & \cdots & \frac{\partial f(x_1)}{\partial Z_{1_N}} \\ \frac{\partial g(x_1)}{\partial X'_1} & \frac{\partial g(x_1)}{\partial Y'_1} & \frac{\partial g(x_1)}{\partial Z'_1} & \frac{\partial g(x_1)}{\partial X_{1_1}} & \frac{\partial g(x_1)}{\partial Y_{1_1}} & \frac{\partial g(x_1)}{\partial Z_{1_1}} & \frac{\partial g(x_1)}{\partial X'_2} & \frac{\partial g(x_1)}{\partial Y'_2} & \cdots & \frac{\partial g(x_1)}{\partial Z_{1_N}} \\ \frac{\partial h(x_1)}{\partial X'_1} & \frac{\partial h(x_1)}{\partial Y'_1} & \frac{\partial h(x_1)}{\partial Z'_1} & \frac{\partial h(x_1)}{\partial X_{1_1}} & \frac{\partial h(x_1)}{\partial Y_{1_1}} & \frac{\partial h(x_1)}{\partial Z_{1_1}} & \frac{\partial h(x_1)}{\partial X'_2} & \frac{\partial h(x_1)}{\partial Y'_2} & \cdots & \frac{\partial h(x_1)}{\partial Z_{1_N}} \\ \frac{\partial f(x_2)}{\partial X'_1} & \frac{\partial f(x_2)}{\partial Y'_1} & \frac{\partial f(x_2)}{\partial Z'_1} & \frac{\partial f(x_2)}{\partial X_{1_1}} & \frac{\partial f(x_2)}{\partial Y_{1_1}} & \frac{\partial f(x_2)}{\partial Z_{1_1}} & \frac{\partial f(x_2)}{\partial X'_2} & \frac{\partial f(x_2)}{\partial Y'_2} & \cdots & \frac{\partial f(x_2)}{\partial Z_{1_N}} \\ \frac{\partial g(x_2)}{\partial X'_1} & \frac{\partial g(x_2)}{\partial Y'_1} & \frac{\partial g(x_2)}{\partial Z'_1} & \frac{\partial g(x_2)}{\partial X_{1_1}} & \frac{\partial g(x_2)}{\partial Y_{1_1}} & \frac{\partial g(x_2)}{\partial Z_{1_1}} & \frac{\partial g(x_2)}{\partial X'_2} & \frac{\partial g(x_2)}{\partial Y'_2} & \cdots & \frac{\partial g(x_2)}{\partial Z_{1_N}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h(x_n)}{\partial X'_1} & \frac{\partial h(x_n)}{\partial Y'_1} & \frac{\partial h(x_n)}{\partial Z'_1} & \frac{\partial h(x_n)}{\partial X_{1_1}} & \frac{\partial h(x_n)}{\partial Y_{1_1}} & \frac{\partial h(x_n)}{\partial Z_{1_1}} & \frac{\partial h(x_n)}{\partial X'_2} & \frac{\partial h(x_n)}{\partial Y'_2} & \cdots & \frac{\partial h(x_n)}{\partial Z_{1_N}} \end{bmatrix}$$



$$= \begin{bmatrix} (1+\Delta S) & -\kappa & \varphi & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \kappa & (1+\Delta S) & -\omega & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -\varphi & \omega & (1+\Delta S) & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1+\Delta S) & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1+\Delta S) & -\kappa & \varphi & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa & (1+\Delta S) & -\omega & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

but, since the angles are small (approaching zero) the matrix becomes

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

GENERALIZED BURSA-WOLF TRANSFORMATION

- Vector of parameters:

$$\mathbf{\Delta} = \begin{bmatrix} T_X & T_Y & T_Z & s & R_\omega & R_\varphi & R_\kappa \end{bmatrix}^{\mathrm{T}}$$

- f-matrix

$$\mathbf{f} = \begin{bmatrix} X - x \\ Y - y \\ Z - z \end{bmatrix}$$

GENERALIZED BURSA-WOLF TRANSFORMATION

- If no common points in both systems – requires observations
  - Additional observation equations

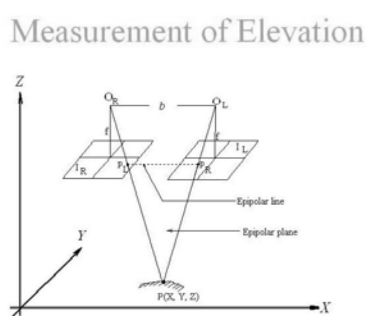
$$\mathbf{F} = \mathbf{M} - \mathbf{S}\left(\mathbf{G} - \mathbf{dx}\right) = \mathbf{0}$$

- Where: M = X, Y, Z coordinates – classical
  G = X, Y, Z coordinates in new datum
  dx = X, Y , Z coordinate difference
  in connecting survey
  S = scale factor

Expanded/Full Model

- Angles not small – non-linear

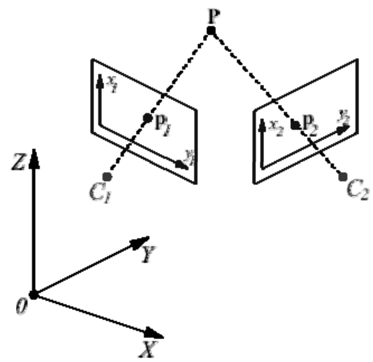
$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = S \begin{bmatrix} \cos\varphi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\varphi\cos\kappa & \sin\omega\sin\kappa - \cos\omega\sin\varphi\cos\kappa \\ -\cos\varphi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\varphi\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\varphi\sin\kappa \\ \sin\varphi & -\sin\omega\cos\varphi & \cos\omega\cos\varphi \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$$



Expanded/Full Model

- Solution:

$$
\begin{aligned}
\mathbf{H} &= \text{estimates} \\
\mathbf{N} &= \mathbf{B}^{T}\mathbf{W}\mathbf{B} \\
\mathbf{t} &= \mathbf{B}^{T}\mathbf{W}\mathbf{f} \\
\boldsymbol{\Delta} &= \mathbf{N}^{-1}\mathbf{t} \\
\mathbf{H}' &= \mathbf{H} + \boldsymbol{\Delta}
\end{aligned}
$$



Generalized Full Model

- Solution: AV + BΔ = f

$$
\begin{bmatrix}
Sr_{11} & Sr_{21} & Sr_{31} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
Sr_{12} & Sr_{22} & Sr_{32} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
Sr_{13} & Sr_{23} & Sr_{33} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Sr_{11} & Sr_{21} & Sr_{31} & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & Sr_{13} & Sr_{23} & Sr_{33} & 0 & \cdots & -1
\end{bmatrix}
\begin{bmatrix}
v_{X_1} \\ v_{Y_1} \\ v_{z_1} \\ v_{X_2} \\ \vdots \\ v_{Z_n}
\end{bmatrix}
$$

$$
+\begin{bmatrix}
b_{11} & 0 & b_{13} & b_{14} & 1 & 0 & 0 \\
b_{21} & b_{22} & b_{23} & b_{24} & 0 & 1 & 0 \\
b_{31} & b_{32} & b_{33} & b_{34} & 0 & 0 & 1 \\
b_{41} & 0 & b_{43} & b_{44} & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{m1} & b_{m2} & b_{m3} & b_{m4} & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
dS \\ dR_{\omega} \\ dR_{\varphi} \\ dR_{\kappa} \\ d\Delta X \\ d\Delta Y \\ d\Delta Z
\end{bmatrix}
= f
$$

Generalized Full Model

$$b_{11} = r_{11}X_1 + r_{21}Y_1 + r_{31}Z_1$$
$$b_{13} = S\left(-X_1 \sin\varphi \cos\kappa + Y_1 \sin\varphi \sin\kappa + Z_1 \cos\varphi\right)$$
$$b_{14} = S\left(r_{21}X_1 - r_{11}Y_1\right)$$

$$b_{21} = r_{12}X_1 + r_{22}Y_1 + r_{32}Z_1$$
$$b_{22} = -S\left(r_{13}X_1 + r_{23}Y_1 + r_{33}Z_1\right)$$
$$b_{23} = S\left(X_1 \sin\omega \cos\varphi \cos\kappa - Y_1 \sin\omega \cos\varphi \sin\kappa + Z_1 \sin\omega \sin\varphi\right)$$
$$b_{24} = S\left(r_{22}X_1 - r_{12}Y_1\right)$$

$$b_{31} = r_{13}X_1 + r_{23}Y_1 + r_{33}Z_1$$
$$b_{32} = S\left(r_{12}X_1 + r_{22}Y_1 + r_{32}Z_1\right)$$
$$b_{33} = S\left(-X_1 \cos\omega \cos\varphi \cos\kappa + Y_1 \cos\omega \cos\varphi \sin\kappa - Z_1 \cos\omega \sin\varphi\right)$$
$$b_{34} = S\left(r_{23}X_1 - r_{13}Y_1\right)$$

Polynomial Approach

- Another approach

$$x' = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8yz + a_9zx + a_{10}xy^2 + a_{11}x^2y + a_{12}xz^2 + \cdots$$

$$y' = b_0 + b_1x + b_2y + b_3z + b_4x^2 + b_5y^2 + b_6z^2 + b_7xy + b_8yz + b_9zx + b_{10}xy^2 + b_{11}x^2y + b_{12}xz^2 + \cdots$$

$$z' = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5y^2 + c_6z^2 + c_7xy + c_8yz + c_9zx + c_{10}xy^2 + c_{11}x^2y + c_{12}xz^2 + \cdots$$

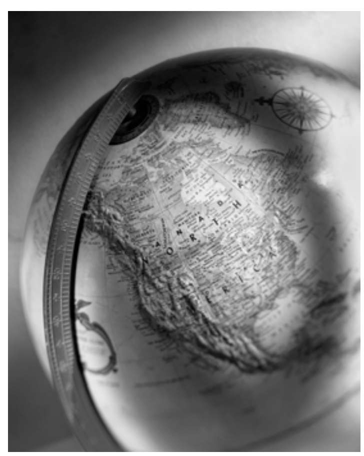
MULTIPLE REGRESSION EQUATIONS (MRE)

NGA Research Grant

Surveying Engineering
Ferris State University

Why MRE Developed?

- Need for better accuracy than available from Molodensky transformation
- Need to simplify procedure for field use
  - “automates” transformation process



WGS 84 Coordinates Obtained Using:

$$\varphi_{WGS84} = \varphi_{LGS} + \Delta\varphi$$
$$\lambda_{WGS84} = \lambda_{LGS} + \Delta\lambda$$
$$H_{WGS84} = H_{LGS} + \Delta H$$

- Where WGS84 relates to the new datum and LGS is the local geodetic system
- Geodetic coordinates (φ, λ, H) referenced to ellipsoid
- Δφ, Δλ, ΔH obtained from MRE transformation

Multiple Regression Equations:

$$\begin{aligned}\Delta\varphi = {} & A_0 + A_1 U + A_2 V + A_3 U^2 + A_4 UV + A_5 V^2 + \cdots + A_{54} V^9 + \\ & A_{55} U^9 V + A_{56} U^8 V^2 + \cdots + A_{64} U^9 V^2 + A_{65} U^8 V^3 + \cdots + \\ & A_{72} U^9 V^3 + A_{73} U^8 V^4 + A_{99} U^9 V^9\end{aligned}$$

- A_i = MRE coefficient – arrived at in stepwise fashion
- U = Normalized geodetic latitude $\quad U = k(\varphi - \varphi_m)$
- V = Normalized geodetic longitude $\quad V = k(\lambda - \lambda_m)$
- k = scale factor and degree-to-radian conversion
- φ, λ = local geodetic latitude and longitude in degrees
- φ_m, λ_m = mid-latitude and mid-longitude of local geodetic area in degrees

Multiple Regression Equations:

$$\Delta\lambda = A_0 + A_1U + A_2V + A_3U^2 + A_4UV + A_5V^2 + \cdots + A_{54}V^9 + A_{55}U^9V + A_{56}U^8V^2 + \cdots + A_{64}U^9V^2 + A_{65}U^8V^3 + \cdots + A_{72}U^9V^3 + A_{73}U^8V^4 + A_{99}U^9V^9$$

$$\Delta H = A_0 + A_1U + A_2V + A_3U^2 + A_4UV + A_5V^2 + \cdots + A_{54}V^9 + A_{55}U^9V + A_{56}U^8V^2 + \cdots + A_{64}U^9V^2 + A_{65}U^8V^3 + \cdots + A_{72}U^9V^3 + A_{73}U^8V^4 + A_{99}U^9V^9$$