# Systems of Equations

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# Linear Equation

- Linear equation in n unknowns in form:

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b$$

- Linear system of m equations in n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

## System of linear equations

- $x_1, ..., x_n$ ==> variables or unknowns
- Givens are numbers $a_{ij}$ and $b_i$
- Each equation represents dot product of the matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

with the column of unknowns to equal b

## EXAMPLE LINEAR SYSTEMS

- 2 x 2 system:

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$$

- 2 x 3 system:

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$$

- 3 x 2 system:

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned}$$

## Linear Equations

- To solve, have to find $x_i$'s
- Solution set: set of all solutions to a linear system
- System of equations consistent if it has at least one solution
  - Nonempty solution set
  - Solution must either have exactly 1 solution or many solutions
- System inconsistent if it has no solution
  - Solution set empty

## Equivalent Systems

- Two systems of equations involving the same variables are equivalent if they have the same solution set
  - If one equation is multiplied by a nonzero real number, this will have no effect on the solution
  - If a multiple of one equation is added to another – new system will be equivalent
  - If we interchange the order of 2 equation, will not affect solution

# Triangular System

- A system is said to be in triangular form if in the $k^{th}$ equation the coefficients of the first k-1 variables are all zero and the coefficient of $x_k$ is nonzero (k=1,...,n)
- Solve using back substitution

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 3 \end{aligned}$$

# Triangular System

- Example:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 2x_4 &= 1 \\ x_2 - 2x_3 + 3x_4 &= 2 \\ 4x_3 + 3x_4 &= 3 \\ 4x_4 &= 4 \end{aligned}$$

# Triangular System

- Solution:

$$\begin{aligned} 4x_4 &= 4 & x_4 &= 1 \\ 4x_3 + (3)(1) &= 3 & x_3 &= 0 \\ x_2 - (2)(0) + (3)(1) &= 2 & x_2 &= -1 \\ 2x_1 - (1)(-1) + (3)(0) - (2)(1) &= 1 & x_1 &= 1 \end{aligned}$$

- Solution: (1, -1, 0, 1)

# Triangular System

- Example:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

- Solution:
  - Place in triangular form
  - Subtract 3 times first row from 2nd gives

    $-7x_2 - 6x_3 = -2$
  - Subtract 2 times first row from 3rd gives
  - $-x_2 - x_3 = -2$

## Triangular System

- System of equations is:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 3 \\
-7x_2 - 6x_3 &= -10 \\
-x_2 - x_3 &= -2
\end{aligned}
$$

  – Multiply 2nd equation by (-1/7) and add to 3rd gives the triangular form

## Triangular System

- Final triangular form:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 3 \\
-7x_2 - 6x_3 &= -10 \\
-\frac{1}{7}x_3 &= -\frac{4}{7}
\end{aligned}
$$

- Using back substitution, the solution set is $x_1$ = 3, $x_2$ = -2, $x_3$ = 4, or more compactly as (3, -2, 4)

## Coefficient Matrix

- System of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

- Arrange coefficients into 3x3 array - coefficient matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

## Augmented Matrix

- Attach to coefficient matrix additional column whose entries are numbers on right hand side of equation, new matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]$$

## Elementary Row Operations

I. Interchange two rows
II. Multiply a row by a nonzero real number
III. Replace a row by its sum with a multiple of another row

## Row Echelon Form

- An (mxn) matrix C is in row echelon form if:

  1) All rows consisting entirely of zeros grouped together at bottom of matrix

  2) First nonzero entry in $(i+1)^{st}$ row appears in column to right of first nonzero entry in $i^{th}$ row

# Row Echelon Form

$$\begin{bmatrix} 2 & 1 & 6 & 4 & -2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Row Echelon Form

- Theorem:
  - Let B be an (mxn) matrix, then there is an (mxn) matrix C such that
    - C is in echelon form
    - B is row equivalent to C

## Gaussian elimination

- Process of using row operations to transform a linear system into one whose augmented matrix is in row echelon form

## Overdetermined Systems

- Linear system where there are more equations than unknowns (m>n)
- Overdetermined systems usually inconsistent

## Underdetermined Systems

- System where there are fewer equations than unknowns ($m<n$)
- Usually consistent with infinitely many solutions

## Reduced Row Echelon Form

- A matrix is in reduced row echelon form if:
  - The matrix is in row echelon form
  - The first nonzero entry in each row is the only nonzero entry in the column
- Gauss-Jordan reduction used to transform into reduced row echelon form

## Reduced Row Echelon Form

- Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix};\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## Homogeneous System

- Equations are homogeneous if the constants on the right-hand side are all zero
- Systems always consistent
- If there is a unique solution, it must be the trivial solution
- An m x n homogeneous system has a nontrivial solution if n > m.