Statistical Tests as Guidelines in Analyses of Adjustment of Control Nets

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ABSTRACT—A minimum constraint solution is suggested as the first step in analyses of an adjustment. If the a posteriori variance of unit weight is found to be too large as compared to the a priori variance of unit weight, there are the following possible causes, which are discussed: mathematical model, computational errors, ill-conditional system, influence of omitted higher order terms, incorrect estimate of a priori variances of observations, and blunders in observations. Influence of a blunder in an observation on various quantities is presented in detail and a method of locating an observation which is subject to a possible blunder is discussed. It is emphasized that three times the standard error of an observation should not be used as a measure of a blunder in adjustments of control nets. A statistical test is presented which can be used for checking whether additional constraints are significantly deforming the minimum constraint solution. Steps are also suggested for locating the problem constraints.

Introduction

Since Gauss and Legendre introduced the principle of the least sum of squares of residuals in 1794 (published 1809) and 1806 respectively, the method has been widely used in various types of problems where there are more observations than would have been needed for a solution. During the following 100–150 years the method was developed further, but usually its utilization had only one objective: to find numerical values for parameters. Parallel to these computational techniques, applications of the statistical methods were developed. The introduction of highspeed computers really changed computational and analysis techniques. The task of solving a set of parameters from field observations using the least-squares technique became routine. Currently it has become more and more important to evaluate observations and computational results for quality and uniformity. All of us are aware that nowadays prior analyses of the planned field work is standard office procedure. Optimization—where expected accuracy and economical considerations have been included—is getting a high priority in any planning today. The following is limited to some analyses of control nets after the field observations have been made.

Computational Technique

In order to simplify the problem, let us assume that the method of observation equations (variation of coordinates) is used in the adjustment. Let L_b matrix represent the field observations, Σ_{L_b} variance-covariance the matrix of observations, and P the weight matrix. There now exists the relation $P = \sigma_o^2 \Sigma^{-1}_{L_b}$ where σ_o^2 is the a priori variance of unit weight (any suitable scalar, for example, equal to 1). Each observed quantity is expressed as a function of the parameters in the model:

$$L_a = F(X_a)$$

where L_a = theoretical values of observed quantities and X_a = theoretical values of parameters (in this case usually coordinates of the control points). Usual Taylor series

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linearization gives the observation equations:

$$V = AX + L$$

where

V = residuals,

$$A = \frac{\partial F}{\partial X_a}, L = L_o - L_b; L_o = F(X_o)$$

and X_o = approximate values of parameters and

$$X = X_a - X_o.$$

The least-squares solution gives us the estimates for X:

$$\hat{X} = -(A'PA)^{-1}A'PL$$

and the variance-covariance matrix of the parameters:

$$\Sigma_{\hat{x}_a} = \sigma_o^2(A'PA)^{-1}$$

and furthermore the corresponding square sum of weighted residuals:

$$\hat{V}'P\hat{V} = L'PL + \hat{X}'A'PL.$$

Let us assume that there are n observations and u parameters, then the expected value, $E(V'PV) = \sigma_o^2(n-u)$ (Hamilton, 1964), from which the a posteriori variance of unit weight is obtained:

$$\hat{\sigma}_o^2 = \frac{\hat{V}'P\hat{V}}{n-u}.$$

Minimum Constraint Solution

After field observations have been received a check should be made to find whether or not the observations are consistent and without blunders. For this reason the first adjustment should be run with minimum fixed information. For example, if only directions or angles have been observed, the coordinates of one point, one azimuth, and one distance, or the coordinates of two points should be fixed: in the case that directions, distances, and azimuths have been measured, then only coordinates of one point should be fixed. There are, of course, several possible combinations and variations of the above. Shortly, it can be said that, without fixing anything, matrix $A'PA$ has dimensions u by u but the rank of the matrix $A'PA = R(A'PA) < u$. The number of the quantities which have to be fixed now is: $u - R(A'PA)$. The fixed quantities must be properly selected. Using the system with minimum fixed quantities $\hat{V}'P\hat{V}$ is obtained, which is equal to the $\hat{V}'P\hat{V}$ to be obtained by using generalized matrices (such as pseudo inverse) without fixing any values. In order to have an appropriate numerical value for $\hat{V}'P\hat{V}$ it might be necessary to go through the usual iteration procedures in the solution of the parameters. If a condition equation type of model or more complicated model is used, then a proper iteration should be made by using, for example, such techniques as suggested by Pope [1972].

Statistical Testing of $\hat{V}'P\hat{V}$

After the $\hat{V}'P\hat{V}$ corresponds to minimum constraints, a statistical test using the so-called χ^2 test should be performed. Practically speaking, if the H_o hypothesis means that everything seems to work properly then: If

$$\frac{\hat{V}'P\hat{V}}{\sigma_o^2} > \chi^2_{DF,\, \alpha/2} \quad \text{or} \quad \frac{\hat{V}'P\hat{V}}{\sigma_o^2} < \chi^2_{DF(1-\alpha/2)}$$

the H_o hypothesis is rejected, otherwise it is not. The $\hat{V}'P\hat{V}$ is obtained from adjustment and σ_o^2 is the a priori variance of unit weight; χ^2 is taken from statistical tables as a function of DF = degree of freedom which was in the adjustment $(n-u)$ and α which is a selected significance level. Often, cases only need to be considered where the $\hat{V}'P\hat{V}$ is too large; therefore the so-called one-tailed test can be used, in which case: If

$$\frac{\hat{V}'P\hat{V}}{\sigma_o^2} > \chi^2_{DF,\, \alpha}$$

the H_o hypothesis is rejected, otherwise it is not. Usually α is taken between 0.1 and 0.01 values and most commonly 0.05. Examples of χ^2 values are given in Table 1.

If χ^2 tables are not available, then F tables can be used, remembering that

$$\chi^2_{DF,\, \alpha} = DF \cdot F_{DF,\, \infty,\, \alpha}.$$

If the hypothesis is rejected even though it is correct, the probability of committing a Type I error is α. If this H_o is not rejected, even though it is wrong, the probability of committing Type II error is β, which requires longer explanation than can be included here [Baarda, 1968a and b]. This presentation is limited to the acceptance that if the H_o hypothesis is not rejected everything seems to work fine. On the other

Table 1

Percentage Points of $\chi^2_{DF,\,P}$ Distribution

P is equal to $(1-\alpha/2)$, $(1-\alpha)$, $\alpha/2$ or α

DF \ P	0.975	0.950	0.050	0.025
1	0.00+	0.00+	3.84	5.02
5	0.83	1.15	11.07	12.83
10	3.25	3.94	18.31	20.84
20	9.59	10.85	31.41	34.17
50	32.36	34.76	67.50	71.42
100	74.22	77.93	124.34	129.56

hand, if H_0 is rejected this serves as a warning that the $\hat{V}'P\hat{V}$ is too large and there is a need to find out what is wrong in the system. More often than not it is a very difficult task to pinpoint the trouble because combinations of several individual sources are possible.

Possible Causes for a Too Large $\hat{V}'P\hat{V}$

When the $\hat{V}'P\hat{V}$ is too large, one first thinks that there is a problem in observations or in weights of observations. It is, of course, clear that the computed residuals in this case are too large and there could be poorly determined weights; however, observations could be as good as they have been estimated to be and there still could be a problem. Among the causes for a too large $\hat{V}'P\hat{V}$ are: mathematical model, computational errors, ill-conditioned system, influence of omitted higher order terms, incorrect estimates of weights, and blunders in observations. Five of the first six causes are discussed briefly and more discussion is devoted to the blunders in the following paragraphs.

Mathematical model

When speaking here about horizontal control nets, the mathematical models have usually been well-established, however some systematic errors could enter into the model. For example if several distance-measuring instruments have been used and if they have not been calibrated against a common standard, the internal inconsistencies show up in the $\hat{V}'P\hat{V}$. Another example of the problem in the mathematical model is the case of condition equations where the spherical excess has been omitted even though large triangles are used. Neglected influences of skew normals, deflection of verticals, and slope corrections are some additional simple examples belonging in this category.

Computational errors

This group includes such errors as wrong evaluations of partial derivatives, sign errors, punching errors, and programming errors. Usually these can be detected by doing computations and punching twice, preferably using different procedures.

Ill-conditioned system

An ill-conditioned system could be detected through computation of so-called condition-numbers [Faddeev, Faddeeva, 1963]. It is caused almost always by the geometry of the network or high correlations between observations. Commonly this can be detected also by an examination of correlation coefficients of the parameters. If notations $N = A'PA$ and $Q_x = N^{-1}$ are used, then the correlation coefficient between the ith and jth parameters is computed, using the following formula:

$$r_{ij} = \frac{q_{ij}}{\sqrt{q_{ii}q_{jj}}}\ ;\ i \neq j$$

where q's are corresponding elements of the weight coefficient matrix Q. If r is close to -1 or $+1$ an ill-conditioned system can be expected. Frequently it is too late to improve this situation in the office, especially when the horizontal control net is in question, except to add some significant figures throughout the computations, including evaluations of partials and the constant vector. A real improvement can only be obtained through additional fieldwork or a change in design of the network.

Influence of omitted higher order terms in the Taylor series expansion

When dealing with a common horizontal control net there are hardly ever any problems in linearization. One way to test the influence of high order terms would be to compute $\hat{V}'P\hat{V}$ two different ways:

1) through a linear model, for example:

$$\hat{V}'P\hat{V} = L'PL + \hat{X}'A'PL$$

2) through a non-linear model: first computing adjusted values of parameters $\hat{X}_a = X_o + \hat{X}$ and inserting these adjusted values into the mathematical model and computing through a non-linear model the adjusted values of observed quantities, $\hat{L}_a = F(\hat{X}_a)$. Using these $\hat{L}_a$ values and the observations, L_b, the second $\hat{V}'P\hat{V}$ can be computed

$$\hat{V}'P\hat{V}_I = (\hat{L}_a - L_b)'P(\hat{L}_a - L_b).$$

If $\hat{V}'P\hat{V}_I$ differs considerably from $\hat{V}'P\hat{V}$ computed through a linear model iterations must be continued or there is an influence of higher order terms which have not been removed. In the latter case more sophisticated iterations should be applied [Pope 1974, Celmiņš 1973].

Incorrect variance-covariance matrix of observed quantities

First of all, it must be recognized that this whole testing can be done only with the assumption that there is a good estimate of Σ_{L_b}. A priori variances of observations should be carefully determined, using appropriate populations and extensive sampling or using results from earlier adjustments where only one kind of observations was used. If a priori variances for the groups and possible confidence intervals for them have been established, it might be justifiable to change values inside the bounds of the confidence intervals. Variances of observations or weights should never be changed arbitrarily. Criteria must be set up prior to the adjustment, except in very unusual cases. The a priori value of variance of unit weight σ_o^2 selected for the adjustment does not have any influence on the statistical test, because the ratio $\hat{V}'P\hat{V}/\sigma_o^2$ is invariant with regard to σ_o^2.

Blunders in Observations

It has been common practice in many offices to blame field observations, when a new extension of control network did not fit to the old network. If all constraints are applied to the first adjustment without examining other alternative sources, it is more often an unjustified assumption. It is also very difficult to find the observations which could have blunders when all constraints are added right away to the adjustment. Even under the suggested minimum constraint adjustment, it is difficult enough to find them. As various authors have pointed out, the influence of a blunder is distributed to many adjusted values of observations or residuals. The influence of changes in observations are as follows:

$$\begin{bmatrix} dL_b \\ dL_a \\ dX_a \\ dV \end{bmatrix} = \begin{bmatrix} P^{-1} \\ AN^{-1}A' \\ N^{-1}A' \\ AN^{-1}A' - P^{-1} \end{bmatrix} PdL_b.$$

The propagation of errors through a linear model easily gives the corresponding weight coefficient matrices:

$$\begin{bmatrix} Q_{L_b} & Q_{L_bL_a} & Q_{L_bX_a} & Q_{L_bV} \\ Q_{L_aL_b} & Q_{L_a} & Q_{L_aX_a} & Q_{L_aV} \\ Q_{X_aL_b} & Q_{X_aL_a} & Q_{X_a} & Q_{X_aV} \\ Q_{VL_b} & Q_{VL_a} & Q_{VX_a} & Q_V \end{bmatrix} =$$

$$\begin{bmatrix} P^{-1} & AN^{-1}A' & AN^{-1} & AN^{-1}A' - P^{-1} \\ AN^{-1}A' & AN^{-1}A' & AN^{-1} & 0 \\ N^{-1}A' & N^{-1}A' & N^{-1} & 0 \\ AN^{-1}A' - P^{-1} & 0 & 0 & P^{-1} - AN^{-1}A' \end{bmatrix}$$

As is known, the variance-covariance matrices are obtained by multiplying the corresponding weight coefficient matrices by the variance of unit weight. It is interesting to note that the correlation between residuals and adjusted values of parameters is equal to zero. Using weight coefficient matrices, the changes in residuals caused by changes in observations can be written as:

$$dV = Q_{VL_b}PdL_b = -Q_VPdL_b.$$

If the observations are uncorrelated (as they are usually assumed to be in horizontal networks) relative changes in V's caused by a change in ith observation, l_i, can be seen in the corresponding ith column of Q_V-matrix. To obtain these changes in V's, the ith column of Q_V is multiplied by $-p_i dl_i$. In order to understand better the changes in V vector, let us write the equation for Q_V, using the above expressions in the following form:

$$Q_V = Q_{L_b} - Q_{L_a} \text{ or } \Sigma_V = \Sigma_{L_b} - \Sigma_{L_a}.$$

It is interesting to note that when variances of adjusted values of observed quantities are decreasing the variances of corresponding residuals are increasing and vice versa. Usually variances of adjusted values of observed quantities are found to be close to corresponding variances of observed values at a weak part of a horizontal net. This means that variances of residuals and residuals themselves are usually relatively small at that same weak part of the net. The size of a residual depends on the value of the observation, but from the above discussions it should be clear that the size of a residual and its variance also depends on the design matrix or, in other words, on the geometry of the network. Therefore, it can be concluded that in a least-squares adjustment, three times the standard error of an observation cannot be used as a criterion to detect blunders. This is especially true if one wishes to use the same level of probability in the detection. It should also be noted that the large degree of freedom does not necessarily make the so-called 3σ test more valid. In order to have a meaningful value for checking blunders, values which are standardized must be computed. Baarda [1968a] has suggested a hypothesis test H_a, which includes non-central quantity. Without going into theory, the computational technique can be very loosely explained as follows: H_a can be tested with each v_i and corresponding σ_{v_i} (which is a square root of ith diagonal element of Σ_V with the a priori variance of unit weight σ_o^2):

If $$\left|\frac{v_i}{\sigma_{v_i}}\right| > F^{\frac{1}{2}}_{1, \infty, 1-\alpha_o}$$

then do not reject H_{ai}, otherwise reject H_{ai}. The H_{ai} hypothesis is that there is a blunder in the ith observation. Baarda [1968b] gives monograms for this F distribution as a function of α_o and β_o values. In his method the Type I and the Type II errors are controlled simultaneously. Some typical F values are given in Table 2 [Baarda 1968b]:

Table 2

F-values for $\beta_o = 0.80$

α_o	$F_{1, \infty, 1-\alpha_o}$	$F^{\frac{1}{2}}_{1, \infty, 1-\alpha_o}$
0.001	10.80	3.29
0.01	6.66	2.58
0.05	3.84	1.96
0.10	2.72	1.65

It is obvious from Table 2 that a blunder should be looked for first in the observation, where the ratio v/σ_v is the largest one. If adjustment can the done without the observation, where the presence of a blunder is suspected, it should be done and the statistical testing described above should be repeated. Of course, several suspected observations could be omitted simultaneously and the testing done, or a whole net with a minimum number of observations could be taken, adding one observation at a time and testing statistically each time in order to find the observations which would give unacceptable increase to $\hat{V}'P\hat{V}$ which would indicate the presence of blunders. At this point it might be necessary to go to the field and remeasure the quantities in order to have replacements for the observations which are expected to have blunders.

Solution With All Constraints

After reaching an acceptable solution with the minimum constraints, then a solution is obtained with all constraints, such as fixing the coordinates of all old control points included in the net in question. As is well-known, coordinates of the old control net might not all be in the same system. This will show up in the new $\hat{V}'P\hat{V}$ which is computed for this combined system. The $\hat{V}'P\hat{V}$ is usually increased by fixing all the old coordinates, which are not consistent because they will twist or otherwise deform the net and increase the values of v's. Now the question arises, how is a decision made that the deformation is insignificant or significant. The most impersonal method is to use a statistical test again. Let us assume that $\hat{V}'P\hat{V}_o$ is obtained with minimum constraint

Table 3

F-statistics for $\alpha = 0.05$

n-u \ b	1	2	3	4
5	6.61	5.79	5.41	5.19
10	4.96	4.10	3.71	3.48
20	4.35	3.49	3.10	2.87
60	4.00	3.15	2.76	2.53
120	3.92	3.07	2.68	2.45
∞	3.84	3.00	2.60	2.37

in which there is $n-u$ degree of freedom and $\hat{V}'P\hat{V}_2$, with additional b new constraints to the minimum constraint net. Again, loosely speaking, the H_o test means that our net is not deformed significantly. Test as follows:

If

$$\frac{\hat{V}'P\hat{V}_2 - \hat{V}'P\hat{V}_o}{\hat{V}'P\hat{V}_o} \cdot \frac{DF}{b} > F_{,\,b,\,n-u,\,\alpha}$$

the H_o hypothesis is rejected; otherwise, it is not. Some sample values of F are given in Table 3. More complete tables can be found in any statistical textbook.

If the H_o hypothesis is rejected, this means that the added constraint was not consistent or, for example, coordinates of the old control points are not in a same system. When rejection is made, the problem can be located by sequentially adding one constraint at a time to a minimum constraint net. In some cases it is possible to locate a problem constraint and in other cases it is not possible to identify but only to recognize that there is a problem in the area.

Conclusion

It has been demonstrated that statistical tests can be used as guidelines in making decisions in the adjustment of horizontal networks. The described tests should be programmed to any least-squares solution, including those solutions offered by commercial computer services. The above discussions are not limited only to least-squares solutions of horizontal networks. The statistical tests given here are valid to any kind of least-squares solution with the same limitations as above. However, it should always be remembered that statistical tests give only guidelines; they are not foolproof.

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