Matrix Algebra
—a tool for engineers and surveyors

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ABSTRACT—Matrix algebra, a mathematical tool which has proven to be extremely useful in modern surveying computations, is neither new or difficult. Prior to the development of the electronic computer, however, matrix algebra was considered too cumbersome for practical use and was therefore not included in engineering and surveying curriculums. As a consequence, there appears to be a general lack of understanding of matrix methods among many practicing professionals whose formal educations terminated more than about a decade ago, which is certainly understandable. This paper bridges this "mathematical generation gap" by presenting an elementary introduction to the subject of matrix algebra with an example of its adaptability and application to the profession of surveying.

INTRODUCTION

Recent technological innovations have produced some truly remarkable changes in the profession of surveying and mapping. These changes have taken place not only in terms of the new equipment now available for taking measurements and gathering data, but also in terms of the electronic equipment currently accessible for processing the data. Modern electronic distance measuring devices, theodolites, and photogrammetric equipment have made both extreme accuracies and high speeds commonplace in surveying and mapping projects of today; and the electronic digital computer has made possible computational techniques which are commensurate with these accuracies and speeds.

There is little doubt that the digital computer represents one of man's greatest technological accomplishments of modern times. Its tremendous capabilities were recently dramatically demonstrated with the placement of astronauts on the moon, a task which hardly could have been accomplished without the computer. Closer to home, however, we surveyors and mappers have also utilized the computer to relieve ourselves of the tedious and time-consuming computational tasks required in our profession. We have realized that certain computational tasks, not considered feasible only a few years ago, have now become practical and popular because of the computer.

With the emergence of the electronic computer, mathematical methods called *matrix algebra* came forth from obscurity. Matrix algebra is not a new mathematical procedure; however, prior to the computer, it was considered too cumbersome for practical utilization. When used in conjunction with a computer, matrix algebra is extremely convenient and efficient. In particular, it affords the following advantages:

1. It provides a convenient and systematic means for storing and manipulating large arrays of numbers.
2. It provides a means of reducing large and complicated systems of equations to simple expressions which can be more easily visualized, manipulated and analyzed.
3. It provides a concise method of expressing algorithms and of directing the computer execution of those algorithms—an especially important aspect where an algorithm is used repeatedly.

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DEFINITIONS

A *matrix* is a set of numbers or symbols arranged in a square or rectangular array of m rows and n columns, and subject to certain rules of operation. An example of matrix representation is the following system of three linear equations involving three unknowns:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= c_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= c_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= c_3 \end{aligned}$$

This generalized system of linear equations may be represented in *matrix form* as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

This, in turn, may be represented in *compact matrix notation* as:

$$\mathbf{A}\,\mathbf{X} = \mathbf{C}$$

In this matrix equation, **A** is the coefficient matrix, **X** is the matrix of unknowns, and **C** is the matrix of constants. Bold face capital letters (as **A, X, C**) are usually used to denote a matrix. From this compact matrix equation it is immediately obvious that matrix methods provide a compact shorthand notation convenient for handling large systems of equations.

The *size, dimension,* or *order of a matrix* is specified by its number of (horizontal) rows m, and its number of (vertical) columns n. Thus the following matrix **B** is a 2 by 3 matrix—there are 2 rows and 3 columns; i.e., $m = 2$, $n = 3$.

$${}_m\mathbf{B}_n = \underset{2\qquad\qquad\quad 3}{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}} = \underset{2\qquad\qquad 3}{\begin{bmatrix} 6 & 2 & 3 \\ 4 & 5 & 4 \end{bmatrix}}$$

Similarily, the following matrix **E** is a 3 by 2 matrix:

$$\mathbf{E} = \underset{3\qquad\qquad 2}{\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \\ e_{31} & e_{32} \end{bmatrix}} = \underset{3\qquad 2}{\begin{bmatrix} 5 & 3 \\ 2 & 4 \\ 7 & 8 \end{bmatrix}}$$

Note that the position of each individual element in a matrix is defined by a double subscript, where the first subscript represents the row and the second represents the column. Note also that the lower case of the letter used to denote the matrix is used to denote any particular element within that matrix. Consequently, e_{32} is in row 3 and column 2 of matrix **E** above and its numerical value is eight. The generalized position of an element of a matrix is given with subscripts ij, jk, etc.

Common Types of Matrices

A *rectangular matrix* is a matrix in which the number of rows is m and the number of columns is n:

$$\mathbf{A} = \underset{2\qquad\qquad 3}{\begin{bmatrix} 6 & 2 & 3 \\ 5 & 5 & 4 \end{bmatrix}}$$

In a *column matrix* the number of rows may be any integer, but the number of columns is one. Matrix **B** is a 3×1 column matrix:

$$\mathbf{B} = \underset{3\qquad 1}{\begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}}$$

In a *row matrix* the number of columns may be any integer, but the number of rows is one. Matrix **C** is a 1×4 row matrix:

$$\mathbf{C} = \underset{1\qquad\qquad\quad 4}{\begin{bmatrix} 7 & -4 & 2 & 8 \end{bmatrix}}$$

In a *square matrix,* the number of rows equals the number of columns:

$$\mathbf{D} = \underset{3\qquad\qquad 3}{\begin{bmatrix} 2 & 4 & -5 \\ -7 & 4 & 3 \\ 2 & -1 & 7 \end{bmatrix}}$$

A *symmetrical matrix* is one which is symmetrical about the main diagonal going from top left to bottom right. In other words, the element of row i and column j equals the element of row j column i. A symmetrical matrix is always a square matrix. Matrix **E** is an example of a 3×3 symmetrical matrix:

$$\mathbf{E} = \underset{3\qquad\qquad 3}{\begin{bmatrix} 6 & -4 & 2 \\ -4 & 3 & 7 \\ 2 & 7 & 5 \end{bmatrix}}$$

In a *diagonal matrix,* only the elements on the main diagonal are not zero. The diagonal matrix is always a square matrix.

Matrix **F** below is a 3 × 3 diagonal matrix:

$$\mathbf{F} = {}_{3}\begin{bmatrix} 8 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3}$$

The *unit matrix* or *identity matrix,* usually symbolized with the letter **I,** is a diagonal matrix whose diagonal elements are *ones.* Matrix **I** below is a 3 × 3 unit matrix:

$$\mathbf{I} = {}_{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3}$$

MANIPULATION OF MATRICES

The Transpose of a Matrix

The *transpose of a matrix* is obtained by interchanging rows and columns; i.e., *row i* of the original matrix becomes *column i* of the transposed matrix. The transposed matrix is denoted by a superscript **T**. It is to be noted that the dimensions of the transpose are the reverse of the dimensions of the original matrix. As an example, the transpose of the example rectangular matrix **A** is:

$$\mathbf{A} = {}_{2}\begin{bmatrix} 6 & 2 & 3 \\ 5 & 5 & 4 \end{bmatrix}_{3} ; \mathbf{T} = {}_{3}\begin{bmatrix} 6 & 5 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}_{2}$$

Addition or Subtraction of Matrices

Matrices must be of the same size (equal dimension) in order to be added or subtracted, and two matrices of equal dimension are said to be *conformable* for addition or subtraction. The sum (or difference) of two matrices is a matrix whose elements are the sum (or difference) of the corresponding elements of the original matrices. Examples of addition and subtraction are:

$$\underset{{}_2\mathbf{A}_3}{\begin{bmatrix} 7 & 2 & -2 \\ 2 & -5 & -4 \end{bmatrix}} + \underset{{}_2\mathbf{B}_3}{\begin{bmatrix} 2 & 1 & 6 \\ -4 & -2 & 7 \end{bmatrix}} = \underset{{}_2\mathbf{C}_3}{\begin{bmatrix} 9 & 3 & 4 \\ -2 & -7 & 3 \end{bmatrix}}$$

$$\underset{{}_2\mathbf{A}_3}{\begin{bmatrix} 7 & 2 & -2 \\ 2 & -5 & -4 \end{bmatrix}} - \underset{{}_2\mathbf{B}_3}{\begin{bmatrix} 2 & 1 & 6 \\ -4 & -2 & 7 \end{bmatrix}} = \underset{{}_2\mathbf{C}_3}{\begin{bmatrix} 5 & 1 & -8 \\ 6 & -3 & -11 \end{bmatrix}}$$

When matrices **A, B,** and **C** are conformable for addition or subtraction, the following statements are true:

a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (Commutative Law)
b) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ (Associative Law)

Multiplication of a Matrix by a Scalar

The product of a scalar (constant) and a matrix is obtained by multiplying each element of the matrix by the scalar. In matrix form, multiplication by a scalar k may be expressed as:

$$k\,{}_4\mathbf{A}_2 = {}_4\mathbf{A}_2\,k = {}_4\mathbf{G}_2$$

The following example illustrates the scalar multiplication procedure:

$$3 \times \begin{bmatrix} 6 & -1 \\ 1 & 5 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 18 & -3 \\ 3 & 15 \\ 9 & 12 \\ 12 & 6 \end{bmatrix}$$

Multiplication of One Matrix by Another

A basic requirement which *must* be satisfied in order to multiply matrix **A** by matrix **B**, (i.e., to find the product **AB** = **C**), is that the number of *columns* in matrix **A** must equal the number of *rows* in matrix **B.** When this condition is satisfied, then **A** and **B** are conformable for multiplication. The product **C** will have the same number of *rows* as **A** and the same number of *columns* as **B.** Thus the following multiplications *are* possible.

$${}_4\mathbf{A}_3\;{}_3\mathbf{B}_3 = {}_4\mathbf{C}_3$$
$${}_3\mathbf{E}_5\;{}_5\mathbf{F}_1 = {}_3\mathbf{G}_1$$

The following multiplications *are not* possible.

$${}_2\mathbf{B}_3\;{}_5\mathbf{A}_2$$
$${}_4\mathbf{A}_2\;{}_4\mathbf{B}_3$$

In matrix multiplication, the element c_{ij} of the product **C** is the sum of the successive products of the elements in *row i* of matrix **A** and the elements in *column j* of matrix **B.** Thus, in multiplying

$${}_2\mathbf{A}_3\;{}_3\mathbf{B}_2 = {}_2\mathbf{C}_2$$
$$\text{element } c_{11} = a_{11}\,b_{11} + a_{12}\,b_{21} + a_{13}\,b_{31}$$

The following numerical example illustrates the multiplication process:

$${}_{2}\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}_{3} \; {}_{3}\begin{bmatrix} 4 & -6 \\ 3 & 3 \\ 2 & -2 \end{bmatrix}_{2} = {}_{2}\begin{bmatrix} 17 & -15 \\ 28 & -22 \end{bmatrix}_{2}$$

Specifically, the individual elements of C are calculated as follows:

$$c_{11} = 2 \times 4 + 1 \times 3 + 3 \times 2 = 17$$
$$c_{12} = 2 \times -6 + 1 \times 3 + 3 \times -2 = -15$$
$$c_{21} = 3 \times 4 + 2 \times 3 + 5 \times 2 = 28$$
$$c_{22} = 3 \times -6 + 2 \times 3 + 5 \times -2 = -22$$

Special note is made of the fact that the product of a unit matrix **I** and any conformable matrix **A** is the original matrix **A**, as illustrated in the following example:

$$_2\mathbf{I}_2\ _2\mathbf{A}_2 = {}_2\mathbf{A}_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}$$

If matrices **A**, **B** and **C** are conformable for multiplication, and in the order indicated below, the following statements are true:

(a) **A (B + C) = A B + A C** (First distributive law)

(b) **(A + B) C = A C + B C** (Second distributive law)

(c) **A (B C) = (A B) C** (Associative law)

The following cautions are also noted:

(d) **A B** is not generally equal to **B A**, and they may not even be conformable. Therefore if two matrices are to be multiplied the order in which the multiplication is to be performed must be specified.

(e) If **A B** = 0, neither **A** nor **B** necessarily equals zero.

(f) If **A B = A C**, matrix **B** does not necessarily equal matrix **C**.

Inverse of a Matrix

DEFINITION

If a matrix is square and non-singular, i.e., if its determinant[1] is not equal to zero, then that matrix possesses an inverse matrix. By definition, the product of the matrix inverse and the original matrix is a unit matrix of dimensions equal to those of the original matrix. Therefore if **A** is a square non-singular matrix, its inverse, denoted $\mathbf{A}^{-1}$, is expressed as:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

[1] The determinant of a square matrix is obtained in the same manner as is the determinant of a square array of numbers in ordinary algebra.

METHODS FOR COMPUTING MATRIX INVERSES

A number of general methods are available for calculating matrix inverses. Two are considered here—the method of adjoints, and the method of elementary row transformations.

The Method of Adjoints

The inverse of **A** found by the method of adjoints is expressed in the following equation:

$$\mathbf{A}^{-1} = \frac{\text{Adjoint of } \mathbf{A}}{\text{Determinant of } \mathbf{A}} = \frac{\text{Adjoint of } \mathbf{A}}{\|\mathbf{A}\|}$$

The adjoint of **A** is the transpose of the matrix obtained by replacing each element of the original matrix by its "signed minor" or *co-factor*. The co-factor of element a_{ij} equals $(-1)^{i+j}$ times the numerical value of the determinant of the remaining array after row i and column j have been eliminated from the matrix.

To illustrate, find the inverse of matrix **A** by the method of adjoints:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

To find the co-factor of a_{11}, strike out row 1 and column 1 of **A**, calculate the determinant of the remaining 2×2 matrix, and then affix the proper sign by multiplying the determinant by $(-1)^{1+1}$. Thus:

$$\text{co-factor of } a_{11} = (-1)^2\,[4 \times 4 - 1 \times 3] = 13$$

Similarly:

$$\text{co-factor of } a_{21} = (-1)^3\,[3 \times 4 - 2 \times 3] = -6$$
$$\text{co-factor of } a_{31} = (-1)^4\,[3 \times 1 - 2 \times 4] = -5$$
$$\text{co-factor of } a_{12} = (-1)^3\,[3 \times 4 - 1 \times 2] = -10$$
$$\text{co-factor of } a_{22} = (-1)^4\,[4 \times 4 - 2 \times 2] = 12$$

etc.

With the co-factors assembled in matrix form, the matrix of co-factors is

$$\begin{bmatrix} 13 & -10 & 1 \\ -6 & 12 & -6 \\ -5 & 2 & 7 \end{bmatrix}$$

To obtain the adjoint, transpose this co-factor matrix; thus

$$\text{Adjoint of } \mathbf{A} = \begin{bmatrix} 13 & -6 & -5 \\ -10 & 12 & 2 \\ 1 & -6 & 7 \end{bmatrix}$$

The determinant of **A** can be found in the usual algebraic manner. Note that the co-factors already obtained in the previous step simplify this computation:

$$\|\mathbf{A}\| = 4\,(13) + 3\,(-6) + 2\,(-5) = 24$$

Finally, the inverse of **A** may now be expressed as:

$$\mathbf{A}^{-1} = \frac{1}{24}\begin{bmatrix} 13 & -6 & -5 \\ -10 & 12 & 2 \\ 1 & -6 & 7 \end{bmatrix} = \begin{bmatrix} 13/24 & -1/4 & -5/24 \\ -5/12 & 1/2 & 1/12 \\ 1/24 & -1/4 & 7/24 \end{bmatrix}$$

The arithmetical work is easily checked since, by definition,

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

$$\frac{1}{24}\begin{bmatrix} 13 & -6 & -5 \\ -10 & 12 & 2 \\ 1 & -6 & 7 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(check)}$$

Inverses by Elementary Transformations

Matrix inverses may be found by elementary row transformations, the most useful transformations being:

a. multiplication (or division) of every element in any row by a non-zero scalar,
b. addition (or subtraction) of the elements in any row to the elements of any other row,
c. combinations of (a) and (b).

If elementary row transformations are successively performed on **A** in such manner that **A** is transformed into **I**, and if throughout that procedure exactly the same row transformations are also performed on the equivalent rows of **I**, then **I** will be transformed into $\mathbf{A}^{-1}$. The procedure is illustrated by the same matrix used to demonstrate the method of adjoints:

Initially, the original matrix and the identity matrix are listed side by side:

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} ; \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The following three row transformations performed on **A** and **I** transform them into matrices $\mathbf{A}_1$ and $\mathbf{I}_1$ respectively:

1. Divide row 1 of matrices **A** and **I** by element a_{11} of matrix **A** (4) and place in row 1 of $\mathbf{A}_1$ and $\mathbf{I}_1$ respectively.
2. Divide row 2 of matrices **A** and **I** by a_{21} (3), subtract from row 1 of matrices $\mathbf{A}_1$ and $\mathbf{I}_1$ and place in row 2 of $\mathbf{A}_1$ and $\mathbf{I}_1$ respectively.
3. Divide row 3 of matrices **A** and **I** by a_{31} (2), subtract from row 1 of matrices $\mathbf{A}_1$ and $\mathbf{I}_1$, and place in row 3 of $\mathbf{A}_1$ and $\mathbf{I}_1$ respectively.

Transformed matrices $\mathbf{A}_1$ and $\mathbf{I}_1$ are:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & -7/12 & 1/6 \\ 0 & -3/4 & -3/2 \end{bmatrix} ; \mathbf{I}_1 = \begin{bmatrix} 1/4 & 0 & 0 \\ 1/4 & -1/3 & 0 \\ 1/4 & 0 & -1/2 \end{bmatrix}$$

Note that the first column of $\mathbf{A}_1$ has now been transformed into the first column of a 3×3 identify matrix. Each succeeding stage will transform an additional column of **A** into the corresponding column of the identity matrix.

Matrices $\mathbf{A}_1$ and $\mathbf{I}_1$ are now transformed into matrices $\mathbf{A}_2$ and $\mathbf{I}_2$ by the following three elementary row transformations:

1. Divide row 2 of $\mathbf{A}_1$ and $\mathbf{I}_1$ by element a_{22} of A_1 (−7/12)—or multiply by −12/7—and place in row 2 of $\mathbf{A}_2$ and $\mathbf{I}_2$.
2. Multiply row 2 of $\mathbf{A}_2$ and $\mathbf{I}_2$ by element a_{12} of A_1 (3/4), subtract from row 1 of $\mathbf{A}_1$ and $\mathbf{I}_1$ and place in row 1 of $\mathbf{A}_2$ and $\mathbf{I}_2$ respectively.
3. Multiply row 2 of $\mathbf{A}_2$ and $\mathbf{I}_2$ by element a_{32} of A_1 (−3/4), subtract from row 3 of $\mathbf{A}_1$ and $\mathbf{I}_1$, and place in row 3 of $\mathbf{A}_2$ and $\mathbf{I}_2$ respectively.

Transformed matrices $\mathbf{A}_2$ and $\mathbf{I}_2$ are now

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 5/7 \\ 0 & 1 & -2/7 \\ 0 & 0 & -12/7 \end{bmatrix} ; \mathbf{I}_2 = \begin{bmatrix} 4/7 & -3/7 & 0 \\ -3/7 & 4/7 & 0 \\ -1/14 & 3/7 & -1/2 \end{bmatrix}$$

Note that the second column of $\mathbf{A}_2$ has now been transformed into column 2 of a 3×3 identity matrix.

Matrices $\mathbf{A}_2$ and $\mathbf{I}_2$ may now be transformed into matrices $\mathbf{A}_3$ and $\mathbf{I}_3$:

1. Divide row 3 of $\mathbf{A}_2$ and $\mathbf{I}_2$ by element a_{33} of matrix A_2 (−12/7), and place in row 3 of $\mathbf{A}_3$ and $\mathbf{I}_3$ respectively.
2. Multiply row 3 of $\mathbf{A}_3$ and $\mathbf{I}_3$ by element a_{13} *of* A_2 *(5/7), subtract from row 1 of* $\mathbf{A}_2$ and $\mathbf{I}_2$, and place in row 1 and $\mathbf{A}_3$ and $\mathbf{I}_3$ respectively.
3. Multiply row 3 of $\mathbf{A}_3$ and $\mathbf{I}_3$ by element a_{23} of A_2 (−2/7), subtract from row 2 of $\mathbf{A}_2$ and $\mathbf{I}_2$ and place in row 2 of $\mathbf{A}_3$ and $\mathbf{I}_3$.

Transformed matrices $\mathbf{A}_3$ and $\mathbf{I}_3$ are:

$$A_3 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$I_3 = A^{-1} = \begin{bmatrix} 13/24 & -1/4 & -5/24 \\ -5/12 & 1/2 & 1/12 \\ 1/24 & -1/4 & 7/24 \end{bmatrix}$$

Notice that by these nine elementary row transformations the original **A** matrix has been transformed into an identity matrix and the original **I** matrix has been transformed into A^{-1}. Note also that A^{-1} obtained by this method agrees exactly with the inverse obtained by the method of adjoints.

It should be fairly obvious that the quantity of work involved in inverting matrices increases vastly with the size of the matrix. It is generally not considered feasible to invert matrices by hand, but rather it should be done on the computer. The procedure of elementary row transformations is very systematic and is ordinarily the procedure for which computers are programmed for finding inverses.

Inverse of a 2×2 matrix

The special case of finding the inverse of a 2×2 matrix is extremely easy. To illustrate, consider the following **A** matrix:

$${}_2A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Its inverse is simply:

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{\|A\|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

This can be readily developed by the method of adjoints. As a simple numerical example let us invert the following **B** matrix;

$$B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}; B^{-1} = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$$

Multiplying B^{-1} **B** yields **I** and verifies the inverse.

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

As stated above, matrix algebra is extremely well adapted to digital computer usage. With only a few systematic programmed matrix steps, large systems of simultaneous linear equations may be almost effortlessly solved on the computer. The key to solving systems of equations by matrix methods is the inverse matrix.

Consider the following matrix representation of a system of linear equations.

$$AX = K \qquad (1)$$

where **A** is the matrix of coefficients
X is the matrix of unknowns
and **K** is the matrix of constant terms.

Pre-multiplying both sides of equation (1) by the inverse of **A** produces the following results:

$$A^{-1}AX = A^{-1}K \qquad (2)$$

By definition $A^{-1}A = I$, the identity matrix; therefore equation (2) may be rewritten as:

$$IX = A^{-1}K \qquad (3)$$

Since **IX** = **X**, drop the **I** of equation (3) and write the following equation for the solution of the unknowns, the **X** matrix:

$$X = A^{-1}K \qquad (4)$$

NUMERICAL EXAMPLES

Example 1

A simple numerical example illustrates the procedure of solving simultaneous linear equations. With a Geodimeter placed at point *A* of Figure 1 and with the reflector placed successively at *B, C,* and *D,* the observed values of *AB, AC,* and *AD* are 125.27, 259.60, and 395.85 meters respecttively. Compute x_1, x_2 and x_3.

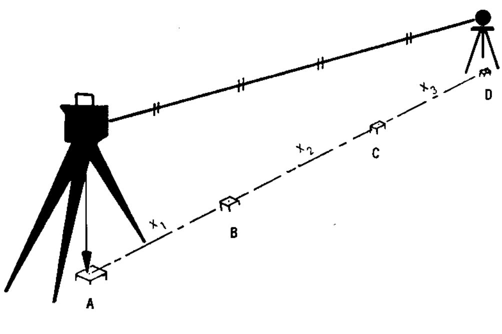


Figure 1

SOLUTION

Formulation of the basic equations and matrix representation:

$$
\begin{aligned}
1\,x_1 + 0\,x_2 + 0\,x_3 &= 125.27\\
1\,x_1 + 1\,x_2 + 0\,x_3 &= 259.60\\
1\,x_1 + 1\,x_2 + 1\,x_3 &= 395.85
\end{aligned}
$$

In matrix notation, these equations are:

$$\mathbf{A\,X} = \mathbf{L}$$

where:

$$\underset{3\ \ 3}{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \underset{3\ \ 1}{\mathbf{X}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \underset{3\ \ 1}{\mathbf{L}} = \begin{bmatrix} 125.27 \\ 259.60 \\ 395.85 \end{bmatrix}$$

The solution in matrix notation is:

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{L}$$

Computation of the Inverse of **A** yields

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution for the unknowns:

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 125.27 \\ 259.60 \\ 395.85 \end{bmatrix} = \begin{bmatrix} 125.27 \\ 134.33 \\ 136.25 \end{bmatrix}$$

Hence

$$
\begin{aligned}
x_1 &= 125.27\\
x_2 &= 134.33\\
x_3 &= 136.25
\end{aligned}
$$

[*Editor's comment*—Obviously this simple problem could be solved easily by elementary arithmetic, but it illustrates clearly how the method may be applied to more involved systems of linear equations.]

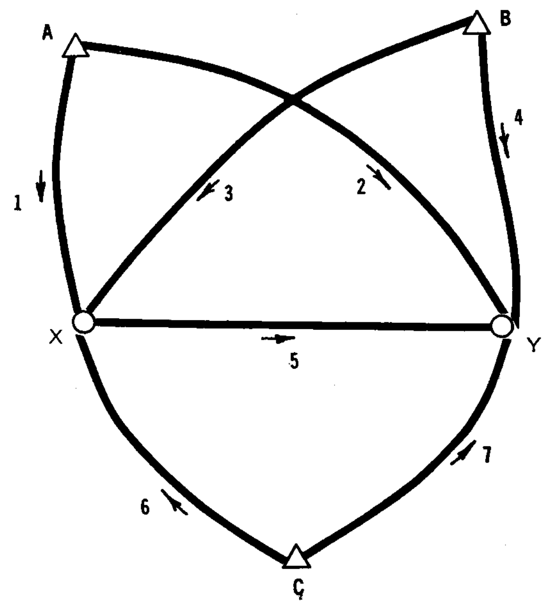
Example 2: Matrix Algorithm

It was mentioned above that matrix algebra is advantageous for simply and concisely expressing and coding algorithms which are used repeatedly. The least squares algorithm, commonly used in surveying, is ideal for illustrating this advantage.

Consider the adjustment of the simple level net of Figure 2. Bench marks exist at *A, B,* and *C,* with elevations 100.00, 101.00, and 102.00 ft above mean sea level, respectively. New field monuments are established at *X* and *Y* and their elevations are obtained by differential leveling along the seven courses shown. Observed differences in elevation and course lengths are given in Figure 2. Adjust this net by least squares.

Formulation of the Observation Equations

Write one observation equation for each course. These equations relate the observed differences in elevation and their inherent



COURSE	LENGTH	ELEV. DIFF.
1	3	-1.00
2	6	-3.05
3	6	-2.05
4	3	-4.00
5	4	-1.95
6	6	-2.95
7	6	-5.05

FIGURE 2

errors to the unknown adjusted values, as follows:

Observation Equations

$$p_1X = p_1\,(A + l_1 + v_1)$$
$$p_2Y = p_2\,(A + l_2 + v_2)$$
$$p_3X = p_3\,(B + l_3 + v_3)$$
$$p_4Y = p_4\,(B + l_4 + v_4)$$
$$p_5Y = p_5\,(X + l_5 + v_5)$$
$$p_6X = p_6\,(C + l_6 + v_6)$$
$$p_7Y = p_7\,(C + l_7 + v_7)$$

In the above observation equations, the p's are relative weights for the courses, (assumed to be inversely proportional to course lengths); X and Y are the unknown adjusted elevations sought; A, B and C are the fixed elevations of the bench marks; the l's are the observed differences in elevation; and the v's are residual errors in the observed values.

Introducing the numerical values for A, B, C and each l, the following simplified equations result:

$$p_1X = p_1\,(99.00 + v_1)$$
$$p_2Y = p_2\,(96.95 + v_2)$$
$$p_3X = p_3\,(98.95 + v_3)$$
$$p_4Y = p_4\,(97.00 + v_4)$$
$$p_5\,(-X + Y) = p_5\,(-1.95 + v_5)$$
$$p_6X = p_6\,(99.05 + v_6)$$
$$p_7Y = p_7\,(96.95 + v_7)$$

Matrix Solution

In matrix form, the above system of equations is represented as:

$$\mathbf{P\,A\,X} = \mathbf{P\,(L + V)} \tag{5}$$

where:

$${}_{7}\mathbf{A}_{2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\quad {}_{2}\mathbf{X}_{1} = \begin{bmatrix} X \\ Y \end{bmatrix},\quad {}_{5}\mathbf{L}_{1} = \begin{bmatrix} 99.00 \\ 96.95 \\ 98.95 \\ 97.00 \\ -1.95 \\ 99.05 \\ 96.95 \end{bmatrix}$$

$$\text{and } {}_{7}\mathbf{V}_{1} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}$$

The **P** matrix is a diagonal matrix with the respective course weights in their respective rows of the matrix. For this example the **P** matrix is:

$${}_{3\frac{1}{2}}\mathbf{P}_{7} = \begin{bmatrix} 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}$$

The matrix algorithm[2] for solving the system of equations by the method of weighted least squares is

$$\mathbf{X} = (\mathbf{A^TPA})^{-1}\,\mathbf{A^TPL} \tag{6}$$

The matrix solution of the algorithm of equation (6) proceeds as follows:

$$\mathbf{A^TP} = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}^{*}$$

$$= \begin{bmatrix} 4 & 0 & 2 & 0 & -3 & 2 & 0 \\ 0 & 2 & 0 & 4 & 3 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A^TPA} = \begin{bmatrix} 4 & 0 & 2 & 0 & -3 & 2 & 0 \\ 0 & 2 & 0 & 4 & 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -3 \\ -3 & 11 \end{bmatrix}$$

$$(\mathbf{A^TPA})^{-1} = \frac{1}{112}\begin{bmatrix} 11 & 3 \\ 3 & 11 \end{bmatrix}$$

$$\mathbf{A^TPL} = \begin{bmatrix} 4 & 0 & 2 & 0 & -3 & 2 & 0 \\ 0 & 2 & 0 & 4 & 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 99.00 \\ 96.95 \\ 98.95 \\ 97.00 \\ -1.95 \\ 99.05 \\ 96.95 \end{bmatrix} = \begin{bmatrix} 797.85 \\ 769.95 \end{bmatrix}$$

$$\mathbf{X} = (\mathbf{A^TPA})^{-1}\,\mathbf{A^TPL} = \frac{1}{112}\begin{bmatrix} 11 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} 797.85 \\ 769.95 \end{bmatrix}$$

$$= \begin{bmatrix} 98.984 \\ 96.991 \end{bmatrix}$$

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[2] Wolf, Paul R., "Horizontal Position Adjustment," SURVEYING AND MAPPING, Vol. XXIX, No. 4, December 1969, pp. 635–644.

* This **P** matrix is the same as **P** above except that it has been multiplied by the scalar 12. Since all weights are relative, this simplifies the computing but does not affect the result.

The adjusted elevations, after rounding are:

$$X = 98.98 \text{ ft above MSL}$$
$$Y = 96.99 \text{ ft above MSL}$$

(The equations necessary for calculating estimated standard deviations of the adjusted elevations are given in footnoted reference No. 2.)

CONCLUSION

The subject of elementary matrix algebra has been introduced and discussed very briefly. Engineering educators are now teaching matrix methods in problem solving as a matter of course; in recent years more and more authors have used matrix algebra in their text books and research articles. These trends, of course, are directly the result of the adaptability of matrix methods to computers. It appears that in the not too distant future a knowledge of matrix algebra will be necessary in order to read most modern texts and research articles. Undoubtedly electronic digital computers and matrix algebra are tools which are here to stay, and their use by engineers and surveyors will expand in years to come. It appears important, therefore, for engineers and surveyors to equip themselves with a knowledge of these subjects.