LU-FACTORIZATION EXAMPLE

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Given the following system of equations:

$$
\begin{aligned}
3x_1 - 4x_2 + 9x_3 - x_4 &= 8 \\
-2x_1 + 6x_2 - x_3 - 7x_4 &= 7 \\
x_2 - x_3 - x_4 &= 0 \\
7x_1 - x_2 + 6x_3 - 5x_4 &= 11
\end{aligned}
$$

Solve the system using LU-Factorization

Begin by computing the upper and lower triangular forms of the design matrix

$$
\begin{bmatrix} 3 & -4 & 9 & -1 \\ -2 & 6 & -1 & -7 \\ 0 & 1 & -1 & -1 \\ 7 & -1 & 6 & -5 \end{bmatrix}
\xrightarrow[R_4 - \frac{7}{3}R_1]{R_2 + \frac{2}{3}R_1}
\begin{bmatrix} 3 & -4 & 9 & -1 \\ 0 & 3.\overline{3} & 5 & -7.\overline{6} \\ 0 & 1 & -1 & -1 \\ 0 & 8.\overline{3} & -15 & -2.\overline{6} \end{bmatrix}
\xrightarrow[R_4 - 2.5R_2]{R_3 - 0.3R_2}
\begin{bmatrix} 3 & -4 & 9 & -1 \\ 0 & 3.\overline{3} & 5 & -7.\overline{6} \\ 0 & 0 & -2.5 & 1.3 \\ 0 & 0 & -27.5 & 16.5\overline{6} \end{bmatrix}
\xrightarrow{R_4 - 11R_3}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.\overline{6} & 1 & 0 & 0 \\ 0 & - & 1 & 0 \\ 2.\overline{3} & - & - & 1 \end{bmatrix}
\qquad
\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.\overline{6} & 1 & 0 & 0 \\ 0 & 0.3 & 1 & 0 \\ 2.\overline{3} & 2.5 & -- & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} 3 & -4 & 9 & -1 \\ 0 & 3.\overline{3} & 5 & -7.\overline{6} \\ 0 & 0 & -2.5 & 1.3 \\ 0 & 0 & 0 & 2.2 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.\overline{6} & 1 & 0 & 0 \\ 0 & 0.3 & 1 & 0 \\ 2.\overline{3} & 2.5 & 11 & 1 \end{bmatrix}
$$

Use the lower triangular form and forward substitution to solve the equations Az=B, where B is the right-hand side the original equations. Thus,

$$z_1 = \frac{b_1}{a_{11}} = \frac{8}{1} = 8$$

$$z_2 = \frac{b_2 - a_{21}z_1}{a_{22}} = \frac{7-\left(-0.\bar{6}\right)8}{1} = 12.\bar{3}$$

$$z_3 = \frac{b_3 - a_{31}z_1 - a_{32}z_2}{a_{33}} = \frac{0-0(8)-0.3\left(12.\bar{3}\right)}{1} = -3.7$$

$$z_4 = \frac{b_4 - a_{41}z_1 - a_{42}z_2 - a_{43}z_3}{a_{44}} = \frac{11-2.\bar{3}(8)-2.5\left(12.\bar{3}\right)-11(-3.7)}{1} = 2.2$$

Thus,

$$z = \begin{bmatrix} 8 \\ 12.\bar{3} \\ -3.7 \\ 2.2 \end{bmatrix}$$

The final process is to use these values for z to solve the given systems of equations by solving the equation Ax = z where A is the upper triangular form of the design matrix. It is found as follows:

$$x_4 = \frac{z_4}{a_{44}} = \frac{2.2}{2.2} = 1$$

$$x_3 = \frac{z_3 - a_{34}x_4}{a_{33}} = \frac{-3.7-1.3\,(1)}{-2.5} = 2$$

$$x_2 = \frac{z_2 - a_{24}x_4 - a_{23}x_3}{a_{22}} = \frac{12.\bar{3}-\left(-7.\bar{6}\right)(1)-5(2)}{3.\bar{3}} = 3$$

$$x_1 = \frac{z_1 - a_{14}x_4 - a_{13}x_3 - a_{12}x_2}{a_{11}} = \frac{8-(-1)(1)-9(2)-(-4)(3)}{3} = 1$$

The solution can then be written as

$$x = \{1 \quad 3 \quad 2 \quad 1\}$$