Subdivision of a Quadrilateral for a Specified Area

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The subdivision of parcels of land is a primary problem of the land surveyor. The subdivision of a quadrilateral (four-sided plane figure) to satisfy simultaneously a specified area and another requirement (i.e., specified frontage, sideline bearing) is a common assignment. The English language surveying literature does not contain a detailed discussion of this problem. Utilizing elementary trigonometrical and geometrical mathematics, this paper presents a set of formulas to subdivide a quadrilateral for any condition and specified area.

La subdivision ou le morcellement des lots de terre demeure un problème important pour l'arpenteur-géomètre. Son travail exige quotidiennement de subdiviser un quadrilatère quelconque (figure planimétrique à quatre côtés) de façon à satisfaire simultanément les caractéristiques géométriques suivantes : soit une superficie déterminée en fonction d'une certaine dimension frontale ou en fonction d'une certaine orientation latérale. La littérature anglaise dans le domaine de l'arpentage ne contient pas une description précise sur ce genre de problème. en faisant appel aux notions trigonométriques et calculs géométriques élémentaires, l'auteur présente un ensemble de formules permettant de subdiviser un quadrilatère d'après toutes sortes de conditions en fonction d'une superficie déterminée.

Introduction

The land surveyor occasionally encounters the problem of subdividing a four-sided (quadrilateral) parcel of land such that one parcel must contain a specified area. The problem is commonly solved by "trial and error" techniques, but it can be solved by exact, noniterative computational techniques. The discussion of this topic, and even the subdivision of other figures, has not been presented in English language surveying literature. Readers of the non-English European surveying literature, which contains more mathematical material, have encountered discussions of these problems.

Figure 1 depicts the problem. Parcel 1 will be the parcel of land having the required area. In this paper, the orientation/relationship of Parcels I and II are as depicted in Figure 1. However, it should be noted that the "orientation" of the quadrilateral is unimportant. To employ the derived equations to a specific problem only requires designation of the corners, azimuths, and distances in the *identical* order shown in Figure 1.

The arrows in the figure indicate the direction of the azimuth along the line. The azimuths are designated $\alpha_1, \alpha_2, \ldots$, and are the *clockwise* angle from north. If the original data are in bearings, the following mathematical relationships will convert the bearings to the required azimuths:

$$\begin{array}{ll} N\ \beta\ E, & \alpha = \beta \\ S\ \beta\ E, & \alpha = 180^\circ - \beta \\ S\ \beta\ W, & \alpha = 180^\circ + \beta \\ N\ \beta\ W, & \alpha = 360^\circ - \beta \end{array}$$

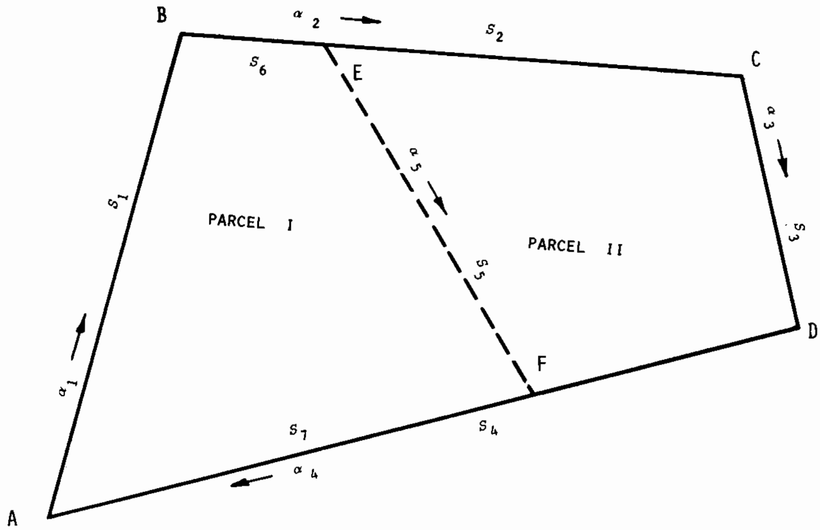

Figure 1

Note that the azimuths for the quadrilateral and Parcel I are in a clockwise manner around each parcel. To ensure that the perimeter dimensions are correct, the coordinates of every corner are computed. These coordinates will be useful in subsequent calculations.

$$x_{i+1} = x_i + s_i \sin \alpha_i \tag{1}$$

$$y_{i+1} = y_i + s_i \cos \alpha_i \tag{2}$$

Where:

x_i, y_i = the coordinates of the ith station.
x_{i+1}, y_{i+1} = the coordinates of the $(i + 1)$ station.
i = the subscript means stations A, B, C, and D in sequence around the quadrilateral.
α_i = azimuth of the line from the ith station *to* the $(i + 1)$ station.
s_i = distance between the ith station and the $(i + 1)$ station.

The quantities ($s_i \sin \alpha_i$) and ($s_i \cos \alpha_i$) will be positive or negative, which results from the algebraic sign associated with the numerical value of the trigonometric functions, and adheres to mathematical laws.

Once the coordinates have been calculated, and the boundary dimensions are verified (or the missing dimensions are calculated), the requisite data is available to calculate the location of the partition. Parcel I in Figure 1 is to contain the specified acreage.

CASE I

Lines *BC* And *AD* Are Parallel
Lines *AB* And *EF* Are Parallel

This is the simplest case. The first step is to calculate the perpendicular distance between the lines *BC* and *AD*. The perpendicular distance, h, is:

$$h = s_1 \sin \theta \tag{3}$$

Where:

θ = the angle at A.

If the required area of Parcel I is K, then:

$$K = s_7 h$$

Substituting equation (3) into this last equation, and solving for the frontage s_7:

$$s_7 = \frac{K}{s_1 \sin \theta} \tag{4}$$

CASE II

Lines BC And AD Are Parallel
The Frontage Is Established (Fixed)

This problem is a trapezoid with the frontage, s_7, given. The altitude of the trapezoid is calculated by equation (3). Then:

$$s_6 = \frac{2K}{h} - s_7 \tag{5}$$

If s_6 in equation (5) is *negative*, then Parcel I will be a *triangle*. Then, the altitude, h, is unknown, and:

$$h = \frac{2K}{s_7} \tag{6}$$

Figure 2 illustrates the problem.

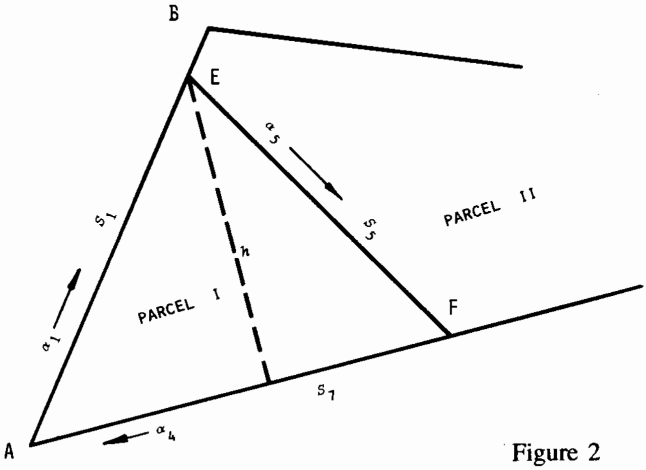

Figure 2

Then, the length of the line AE is:

$$AE = \frac{2K}{s_7 \sin \theta} \tag{7}$$

The length and azimuth of the line EF can be calculated by the inverse coordinate computation or by missing elements of a closed polygon [*Stoughton* 1975].

CASE III

Lines *BC* And *AD* Are Not Parallel
Lines *AB* And *FE* Are Parallel

This problem is the more typical problem, and can be solved by two general procedures. The first procedure is similar to the procedures previously discussed. However, depending on the general "shape" of Parcel I, there are four different solutions. These solutions are based on the location of the *two* acute interior angles. The four solutions are:

IIIa. acute angles at *A* and *B*.
IIIb. acute angles at *A* and *E*.
IIIc. acute angles at *B* and *F*.
IIId. acute angles at *E* and *F*.

The mathematical solution for Case IIIa will be derived in its entirety, and the necessary formulas to solve the other cases will be given.

Case IIIa. Acute angles at points *A* and *B*.

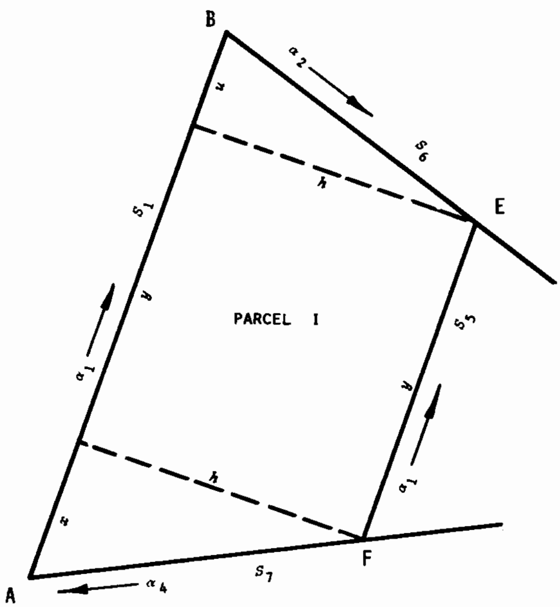

Figure 3

Figure 3 depicts the condition. h is the altitude of the trapezoid. Now:

$$s_1 = x + y + u \tag{8}$$

$$s_5 = y$$

$$\theta_1 = (\alpha_4 \pm 180°) - \alpha_1 \tag{9}$$

$$\theta_2 = (\alpha_1 \pm 180°) - \alpha_2 \tag{10}$$

θ_1 and θ_2 are the acute angles at points A and B respectively. From trigonometry:

$$\tan \theta_1 = \frac{h}{x} \tag{11}$$

$$\tan \theta_2 = \frac{h}{u} \tag{12}$$

Substituting equations (11) and (12) into equation (8) yields:

$$y = s_1 - h \cot \theta_1 - h \cot \theta_2 \tag{13}$$

The area of a trapezoid (employing the notation in Figure 3) is:

$$K = \frac{h}{2}(s_1 + s_5)$$

But, s_5 equals y. Therefore, substituting equation (13) into the area formula yields:

$$K = \frac{h}{2}[2s_1 - h(\cot \theta_1 + \cot \theta_2)]$$

Rearranging:

$$\frac{h^2}{2}(\cot \theta_1 + \cot \theta_2) - hs_1 + K = 0$$

Solving for h (by the quadratic formula):

$$h = \frac{s_1 \pm \sqrt{s_1^2 - 2(\cot \theta_1 + \cot \theta_2)K}}{(\cot \theta_1 + \cot \theta_2)} \tag{14}$$

Case IIIb. Acute angles at points A and E.

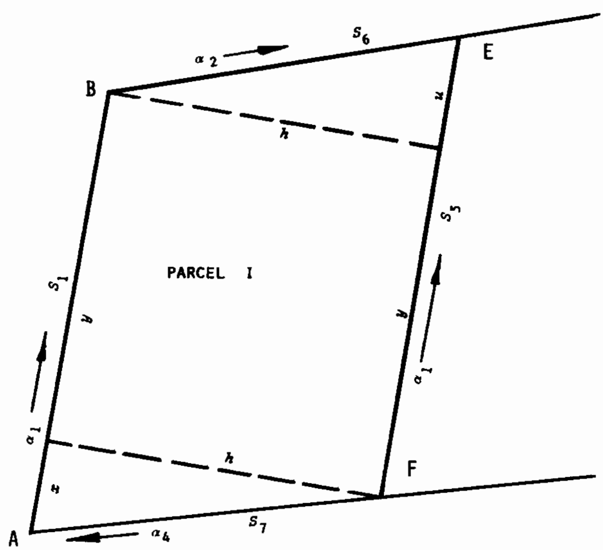

Figure 4

Figure 4 depicts the condition. h is the altitude of the trapezoid.

$$s_1 = x + y$$

$$s_5 = u + y$$

$$\theta_1 = (\alpha_4 \pm 180°) - \alpha_1 \quad (15)$$

$$\theta_2 = \alpha_2 - \alpha_1 \quad (16)$$

Then, the altitude, h, is:

$$h = \frac{-s_1 \pm \sqrt{s_1^2 + 2(\cot\theta_2 - \cot\theta_1)K}}{(\cot\theta_2 - \cot\theta_1)} \quad (17)$$

Case IIIc. Acute angles at points B and F.

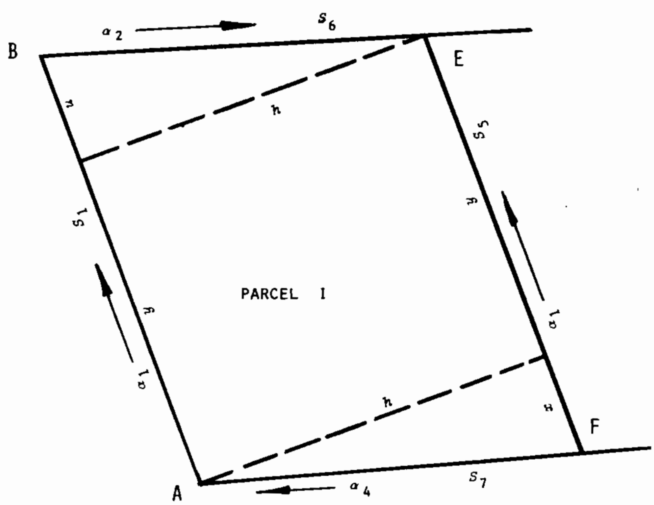

Figure 5

Figure 5 depicts the condition. h is the altitude of the trapezoid.

$$s_1 = y + u$$

$$s_5 = y + x$$

$$\theta_1 = \alpha_1 - \alpha_4 \quad (18)$$

$$\theta_2 = (\alpha_1 \pm 180°) - \alpha_2 \quad (19)$$

Then, the altitude, h, is:

$$h = \frac{-s_1 \pm \sqrt{s_1^2 + 2(\cot\theta_1 - \cot\theta_2)K}}{(\cot\theta_1 - \cot\theta_2)} \quad (20)$$

Case IIId. Acute angles at points E and F.

Figure 6 depicts the condition. h is the altitude of the trapezoid.

$$s_1 = y$$

$$s_5 = x + y + u$$

$$\theta_1 = \alpha_1 - \alpha_4 \tag{21}$$

$$\theta_2 = \alpha_2 - \alpha_1 \tag{22}$$

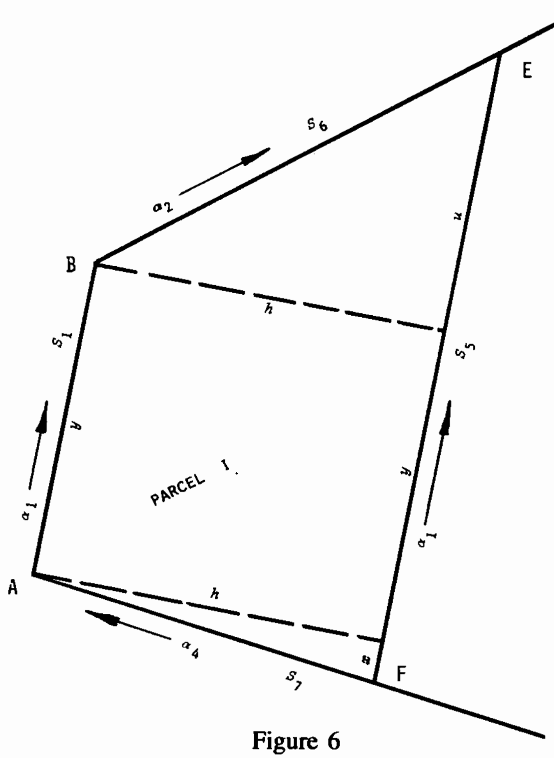

Figure 6

Then, the altitude, h, is:

$$h = \frac{-s_1 \pm \sqrt{s_1^2 + 2(\cot\theta_1 + \cot\theta_2)K}}{(\cot\theta_1 + \cot\theta_2)} \tag{23}$$

After solving for h [equations (14), (17), (20), or (23)], the values of x and u are calculated by equations (11) and (12) respectively. Then, by inspection, s_5 is determined. The lengths s_6 and s_7 can be calculated by employing the following:

$$s_6 = h \csc\theta_2 \tag{24}$$

$$s_7 = h \csc\theta_1 \tag{25}$$

Alternate Computation Procedure

If coordinates are employed, Case III is easily solved by an alternate procedure. There is a single procedure, but it can be divided into two parts. In both instances the procedure is to extend lines *BC* and *AD* to their point of intersection. After the coordinates of the point of intersection are determined, the area of the resulting triangle whose base is *AB* is calculated. A second triangle whose base is *EF* will contain the area of the first triangle with the area of Parcel I being added (or subtracted). The lengths of the base, *EF*, and the altitude are computed from similar triangles.

The first type of problem is where the line *EF* lies between the line *AB* and the point of intersection. Figure 7 depicts the condition.

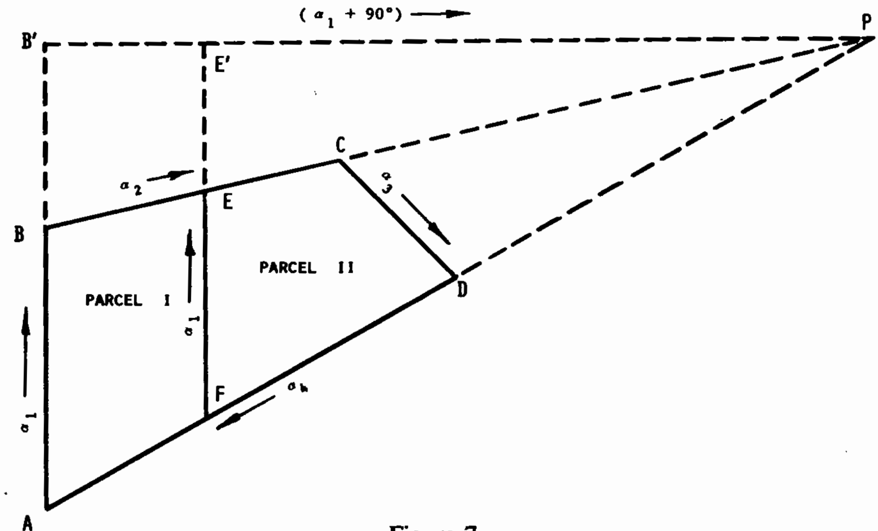

Figure 7

The first step is to calculate the coordinates of the point of intersection, *P*. Let:

$$\theta = \alpha_4 - \alpha_2 \tag{26}$$

$$\Delta x = X_A - X_B \tag{27}$$

$$\Delta y = Y_A - Y_B \tag{28}$$

Note: the subscripts on the coordinates *X* and *Y* refer to the point [*A* and *B*]. Then, the distances *BP* (called s_i) and *PA* (called s_j) are:

$$s_i = \frac{\Delta x \cos \alpha_4 - \Delta y \sin \alpha_4}{-\sin \theta} \tag{29}$$

$$s_j = \frac{\Delta x \cos \alpha_2 - \Delta y \sin \alpha_2}{\sin \theta} \tag{30}$$

Then, the coordinates of point *P* are:

$$X_P = X_B + s_i \sin \alpha_2 \tag{31}$$

$$Y_P = Y_B + s_i \cos \alpha_2 \tag{32}$$

And an independent check on the computations is:

$$X_A = X_P + s_j \sin \alpha_4$$

$$Y_A = Y_P + s_j \cos \alpha_4$$

Next, calculate the altitude, *h*, of the triangle *ABP* [line *B'P* in Figure 7].

$$\Delta x = X_P - X_A \tag{33}$$

$$\Delta y = Y_P - X_A \tag{34}$$

$$h = \Delta x \cos \alpha_1 - \Delta y \sin \alpha_1 \tag{35}$$

The area of the triangle *ABP* is:

$$K_T = \frac{1}{2} s_1 h$$

The area of the triangle *FEP* is:

$$K_R = K_T - K \tag{36}$$

Where:

K_T = the area of triangle *FEP*.
K = the area of Parcel I.

Then, by similar triangles:

$$\mu = \frac{K_R}{K_T} \tag{37}$$

But, by similar triangles:

$$\frac{\mu}{2} s_1 h = \frac{1}{2} s_5 h' \tag{38}$$

Where:
h' = the altitude of triangle *FEP*.

Finally:

$$h' = \sqrt{\mu}\, h \tag{39}$$

$$s_5 = \sqrt{\mu}\, s_1 \tag{40}$$

The lengths s_6 and s_7 can be calculated by the missing elements of a closed polygon procedure.

The second type of problem is where the line *AB* lies between the line *EF* and the point of intersection, *P* (Figure 8).

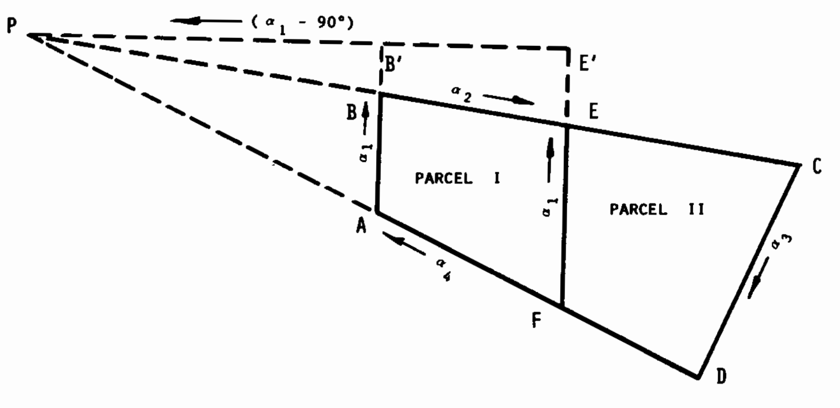

Figure 8

The first step is to calculate the coordinates of the point of intersection. Let:

$$\theta = (\alpha_4 \pm 180°) - (\alpha_2 \pm 180°) \tag{41}$$

$$\Delta x = X_A - X_B \tag{27}$$

$$\Delta y = Y_A - Y_B \tag{28}$$

Then, the distances BP (called s_i) and PA (called s_j) are:

$$s_i = \frac{\Delta x \cos \alpha_4 - \Delta y \sin \alpha_4}{\sin \theta} \tag{42}$$

$$s_j = \frac{\Delta x \cos \alpha_2 - \Delta y \sin \alpha_2}{-\sin \theta} \tag{43}$$

Then, the coordinates of P are:

$$X_P = X_B - s_i \sin \alpha_2 \tag{44}$$

$$Y_P = Y_P - s_i \cos \alpha_2 \tag{45}$$

And an independent check on the computation is:

$$X_A = X_P - s_j \sin \alpha_4$$

$$Y_A = Y_P - s_j \cos \alpha_4$$

Next, calculate the altitude, h, of the triangle APB [line $B'P$ in Figure 8].

$$\Delta x = X_P - X_A \tag{33}$$

$$\Delta y = Y_P - Y_A \tag{34}$$

$$h = -\Delta x \cos \alpha_1 + \Delta y \sin \alpha_1 \tag{46}$$

The area of the triangle FPE is:

$$K_R = K_T + K \tag{47}$$

The remainder of the computational procedure to determine s_5 is performed employing equations (37), (39), and (40).

CASE IV

Lines *BC* And *AD* Are Not Parallel
The Azimuth Of Line *EF* Is Fixed

This fourth problem is the most difficult problem to solve. The first step is to determine the boundary for a parcel with the same area employing Case III. This parcel is $ABE'F'$. Once the dimensions of $ABE'F'$ have been determined, the procedure is to determine the location of the intersection of the line EF and $E'F'$ such that the area of triangle $EE'G$ equals the area of triangle $FF'G$. Figure 9 depicts the problem.

The first step is to calculate the interior angles of the two triangles. Figure 9 depicts the two possible relationships of the Case III solution (dotted line) with respect to the line whose azimuth is given. The formulas necessary to calculate these interior angles are:

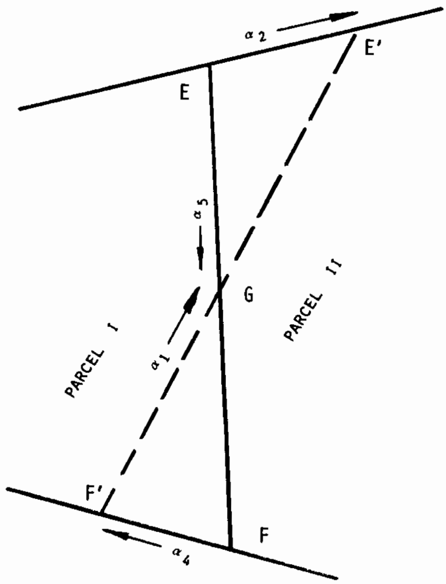

Figure 9(a)

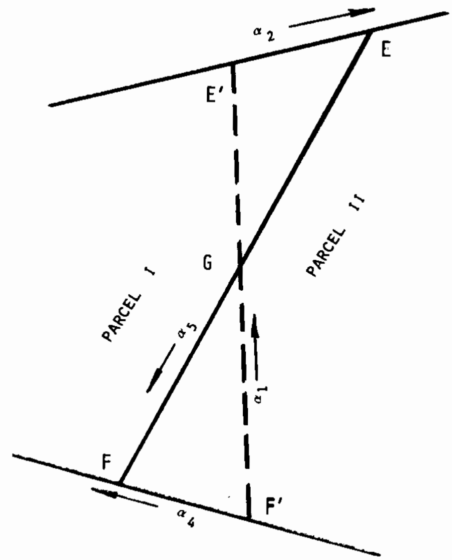

Figure 9(b)

In triangle *GE'E* (Figure 9a):

$\theta_1 = \alpha_1 - (\alpha_5 \pm 180^\circ)$ [angle at G]

$\theta_2 = \alpha_2 - \alpha_1$ [angle at E']

$\theta_3 = \alpha_5 - \alpha_2$ [angle at E]

In triangle *GE'E* (Figure 9b):

$\theta_1 = (\alpha_5 \pm 180^\circ) - \alpha_1$ [angle at G]

$\theta_2 = (\alpha_1 \pm 180^\circ) - \alpha_2$ [angle at E']

$\theta_3 = (\alpha_2 \pm 180^\circ) - \alpha_5$ [angle at E]

In triangle *GFF'* (Figure 9a):

$\phi_1 = (\alpha_1 \pm 180^\circ) - \alpha_5$ [angle at G]

$\phi_2 = (\alpha_4 \pm 180^\circ) - \alpha_1$ [angle at F']

$\phi_3 = (\alpha_5 \pm 180^\circ) - \alpha_4$ [angle at F]

In triangle *GFF'* (Figure 9b):

$\phi_1 = \alpha_5 - (\alpha_1 \pm 180^\circ)$ [angle at G]

$\phi_2 = \alpha_1 - \alpha_4$ [angle at F']

$\phi_3 = \alpha_4 - \alpha_5$ [angle at F]

Note: All the angles calculated above must be less than 180°, and the sum of the three angles must equal 180°.

The length of the line $E'G$ is designated u. Then, the area of the triangle $GE'E$ is:

$$K_1 = \frac{u^2}{2}\frac{\sin\theta_1 \sin\theta_2}{\sin\theta_3} \tag{48}$$

The length of the line $F'G$ is the difference in the length of $E'F'$ (designated z), which was calculated by Class III procedures, and u. Then, the area of triangle $GF'F$ is:

$$K_2 = \frac{(z-u)^2}{2}\frac{\sin\phi_1 \sin\phi_2}{\sin\phi_3} \tag{49}$$

But:

$$K_1 \equiv K_2$$

Therefore:

$$u^2\frac{\sin\theta_1 \sin\theta_2}{\sin\theta_3} = (z-u)^2\frac{\sin\phi_1 \sin\phi_2}{\sin\phi_3} \tag{50}$$

But:

$$\theta_1 = \phi_1$$

Therefore, equation (50) becomes:

$$u^2\frac{\sin\theta_2}{\sin\theta_3} = (z-u)^2\frac{\sin\phi_2}{\sin\phi_3}$$

Expanding and rearranging:

$$\left[\frac{\sin\theta_2}{\sin\theta_3} - \frac{\sin\phi_2}{\sin\phi_3}\right]u^2 + 2z\frac{\sin\phi_2}{\sin\phi_3}u - \frac{\sin\phi_2}{\sin\phi_3}z^2 = 0$$

To simplify, let:

$$U = \frac{\sin\theta_2}{\sin\theta_3} \tag{51}$$

$$V = \frac{\sin\phi_2}{\sin\phi_3} \tag{52}$$

Then, the last unnumbered equation can be written as:

$$(U - V)u^2 + 2zVu - Vz^2 = 0$$

Solving for u (employing the quadratic formula):

$$u = \frac{-V \pm \sqrt{UV}}{(U - V)}z \tag{53}$$

Then, the lengths of the sides of the triangles are:

$$GE = u\frac{\sin\theta_2}{\sin\theta_3} \tag{54}$$

$$EE' = u\frac{\sin\theta_1}{\sin\theta_3} \tag{55}$$

$$GF = (z - u)\frac{\sin \phi_2}{\sin \phi_3} \qquad (56)$$

$$FF' = (z - u)\frac{\sin \phi_1}{\sin \phi_3} \qquad (57)$$

$$s_5 = GE + GF \qquad (58)$$

And, by inspection:

$$s_6 = BE' \pm EE'$$

$$s_7 = AF' \pm FF'$$

Note: in the last two equations, one formula will use addition, and the other formula will employ subtraction. Inspection of Figures 9a and 9b will indicate the choice.

CASE V

Lines *BC* And *AD* Are Not Parallel
The Distance *AF* Is Held (Fixed)

This problem is simpler than the Case IV. The first step is to calculate the area of triangle *ABF* (Figure 10).

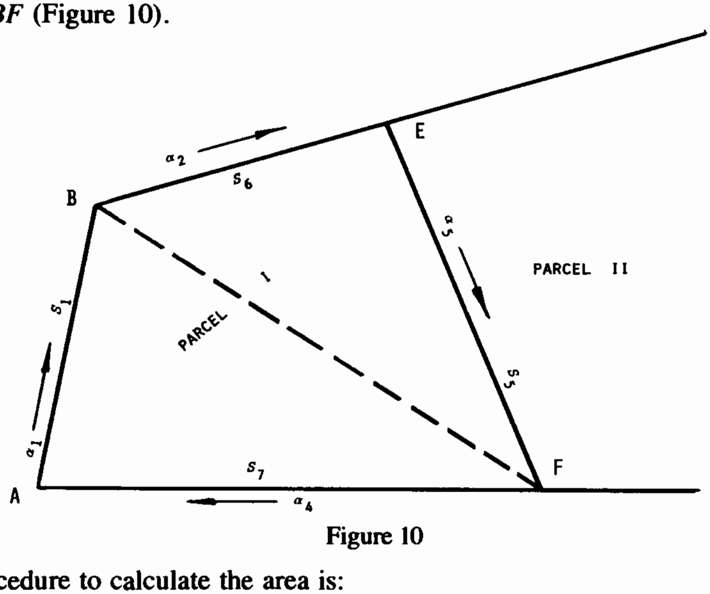

Figure 10

The procedure to calculate the area is:

$$\theta = (\alpha_4 \pm 180°) - \alpha_1 \qquad (59)$$

$$h_1 = s_1 \sin \theta \qquad (60)$$

$$K_1 = \frac{h_1 s_7}{2} \qquad (61)$$

Next, the perpendicular distance from *F* to the line *BE* is calculated [the altitude of triangle *BEF*] by the following formulas (from coordinates):

$$\Delta x = X_F - X_B \qquad (62)$$

$$\Delta y = Y_F - Y_B \tag{63}$$

$$h_2 = \Delta x \cos \alpha_2 - \Delta y \sin \alpha_2 \tag{64}$$

The area of triangle *BEF* must contain the necessary area, when added to the area of triangle *ABF*, the total area will equal the required area for Parcel I. Then:

$$K_2 = K - K_1$$

Where:
K = the required area for Parcel I.
K_2 = the area for triangle *BEF*.

Then, the length of the line *BE* is:

$$s_6 = \frac{2K_2}{h_2} \tag{65}$$

If the area K_1 is larger than the required area for Parcel I, then this problem reduces to the special problem described in Case II.

CASE VI

Lines *BC* And *AD* Are Not Parallel
The Distance *BE* Is Held (Fixed)

This problem is a variation of the previous problem. The method employed follows the same procedure. Figure 11 depicts the problem.

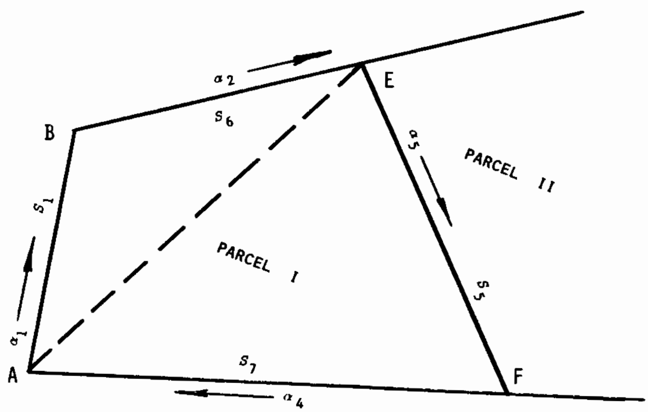

Figure 11

As in Case V, the first step is to calculate the area of a triangle (triangle *ABE*). The procedure is:

$$\theta = (\alpha_1 \pm 180°) - \alpha_2 \tag{66}$$

$$h_1 = s_1 \sin \theta \tag{60}$$

$$K_1 = \frac{s_6 h_1}{2} \tag{67}$$

Then, from coordinates:

$$\Delta x = X_E - X_A \tag{68}$$

$$\Delta y = Y_E - Y_A \tag{69}$$

$$h_2 = \Delta y \sin(\alpha_4 \pm 180°) - \Delta x \cos(\alpha_4 \pm 180°) \tag{70}$$

Then:

$$K_2 = K - K_1$$

And:

$$s_7 = \frac{2K_2}{h_2} \tag{71}$$

The remaining computations can be performed employing the missing elements of a polygon procedure or coordinates.

Conclusion

This paper contains the solutions to the problem of subdividing a quadrilateral (four-sided plane figure) into two parcels with one parcel containing a specified area and satisfying another requirement. The formulas presented utilize well known trigonometrical and geometrical principles. Although this problem has been discussed thoroughly in the non-English European technical literature, it has been disregarded in the English language surveying literature. This is probably due to the general de-emphasis in applied mathematics and surveying computations.

This paper has addressed the subdividion of a quadrilateral, which is a popular "registration/licensure" examination question. Although this paper pertains to the subdivision of a quadrilateral, the methodology (not necessarily the formulas) can be employed to calculate the boundary dimensions of a five-, or more, sided plane figure.

Reference and Bibliography

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Stoughton, H. W. 1975. Computing missing elements of a polygon, *Surveying and Mapping*, Vol. XXXV, No. 3, pp. 217–222.