A General Solution for Obtaining the Missing Element in Plane Closes

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Abstract

A single equation is developed from which two or three missing elements (bearings or distances) of a close may be computed using matrix methods. Some special cases giving rise to a singular matrix are mentioned.

Introduction

Almost all textbooks on surveying give detailed instruction for the computation of two missing elements of a close: a missing line, two missing distances, two missing bearings, or a bearing missing from one line and a distance from another. These computations rely, implicitly at least, on the fact that two unknowns require two equations for solution, and the two equations in this case are:

$$\sum_{i=1}^{n} D_i = 0 \qquad (1)$$

and

$$\sum_{i=1}^{n} L_i = 0 \qquad (2)$$

where D_i = $l_i \text{ Sin } \theta_i$, the departure of line i

L_i = $L_i \text{ Cos } \theta_i$, the latitude of line i

l_i = length of line i

θ_i = bearing of line i

l_i and θ_i both plane since we are dealing only with plane surveying.

Less well publicised is the computation of three elements missing from a close the area of which is known. This is not to say that surveyors are ignorant of the computation. On the contrary, few survey offices would not have a computer programme for such cases, and Capon (1982), for example, gives programming instruction for two such cases — a missing line and distance, and three missing distances. But this author has not seen in any of the usual textbooks on surveying, a general solution for all cases of two or three missing elements.

The introduction of a third missing element requires a third equation in addition to the two equations of closure already given. This third equation is, of course, the area equation.

Area Equation

Areas of closes (polygons) may be computed either by the method of double longitudes or, if they are known, by co-ordinates. This latter method is usually

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remembered by writing out in a row all the East co-ordinates, ending with the first written out again; writing the North co-ordinate below its partner; and then taking the difference of their diagonal products:

$$\begin{array}{ccccccc} E_1 & E_2 & E_{i-1} & E_i & E_{i+1} & E_n & E_1 \\ \\ N_1 & N_2 & N_{i-1} & N_i & N_{i+1} & N_n & N_1 \end{array}$$

i.e.

Double area = sum of "full arrow products"

− sum of "dotted arrow products",

or more generally,

$$DA = \left| \sum_{i=1}^{n} (E_i N_{n+1} - N_i E_{i+1}) \right|$$

where $i = n+1 = 1$.

The contribution, or more loosely the "area" of line i is

$$(E_i N_{i+1} - N_i E_{i+1}).$$

But $E_i = E_1 + \sum_{h=1}^{i} D_h$

$$N_i = N_i + \sum_{h=1}^{i} L_h.$$

Translating the origin of co-ordinates to station 1, and denoting, for the time being the co-ordinates referred to the new origin as E_i', N_i', we have:

$$E' = 0$$

$$N' = 0$$

$$E_i' = \sum_{h=1}^{i} D_i$$

$$N_i' = \sum_{h=1}^{i} L_i$$

and the contribution to the double area made by the i^{th} line of the close is

$$\sum_{h=1}^{i} D_n \sum_{h=1}^{i+1} L_h - \sum_{h=1}^{i} L_h{}^{*} \sum_{h=1}^{i+1} D_h$$

$$= \sum_{h=1}^{i} D_h \left[\sum_{h=1}^{i} L_h + L_i \right] - \sum_{h=1}^{i} L_h \left[\sum_{h=1}^{i} D_h + D_i \right]$$

$$= L_i \sum_{h=1}^{i} D_n - D_i \sum_{h=1}^{i} L_h \qquad (3)$$

Hence the area equation being sought is:

$$DA = \left| \sum_{h=1}^{n} \left[L_i \sum_{h=1}^{i} D_h - D_i \sum_{h=1}^{i} L_h \right] \right| \qquad (4)$$

The Solution

Solution equations (1), (2) and (4) will allow the determination of three unknown elements of a close. But the equations are non-linear, and generally rather "messy" to solve. Numerically, a solution can be found by assuming approximate values for the unknowns and then iteratively finding a correction to those approximate values.

Suppose that in a close the lines j, k and m (in that order) have elements missing but that approximate values are known. Then the departures and latitudes are given by:

$$D_j = D'_j + \Delta D_j \; ; D_k = D'_k + \Delta D_k \; ; D_m = D'_m + \Delta D_m$$

$$L_j = L'_j + \Delta L_j \; ; L_k = L'_k + \Delta L_k \; ; L_m = L'_m + \Delta L_m$$

where primes indicate approximate value, and Δ the respective corrections.

Equations (1) and (2) become:

$$\Delta D_j + \Delta D_k + \Delta D_m = K_1 \qquad (5)$$

$$\Delta L_j + AL_k + \Delta L_m = K_2 \qquad (6)$$

where

$$K_1 = - \sum_{i=1}^{n} l_i \operatorname{Sin} \theta_i,$$ i.e. the correction for the misclosure in departures

and

$$K_2 = - \sum_{i=1}^{n} l_i \operatorname{Cos} \theta_i,$$ the correction for misclosure in latitudes,

the misclose in each case being due to the incorporation in the summation of approximate values for lines j, k and m.

The use of approximate values will also give rise to a misclose in area, the misclose being the difference between that required and that computed with the approximate values.

The use of approximate values for a line affects the computation of the double area in two ways: (a) by the "area" of the line itself, and (b) by the "area" of all subsequent lines. For the jth line (the first with a missing element) the contribution to the double area is

$$\left[L'_j + \Delta L_j \right] \left[\sum_{h=1}^{i} D_h + \Delta D_j \right] - \left[D'_j + \Delta D_j \right] \left[\sum_{h=1}^{i} L_h + \Delta L_j \right]$$

i.e.

$$\left[L'_j \sum^{j} D_h - D'_j \sum^{j} L_h\right] - \Delta D_j\left[\sum^{j} L_h - L'_j\right] + \Delta L_j\left[\sum^{j} D_h - D'_j\right]$$

The first term is the "normal" contribution but computed using approximate values; the second and third terms represent the "misclose" due to those approximations.

For the i^{th} line, $i > j$, the contribution is

$$L_i\left[\sum_{h=1}^{i} D_h + \Delta D_j\right] - D_i\left[\sum_{h=1}^{i} L_h + \Delta L_j\right]$$

$$= L_i \sum_{h=1}^{i} D_h - D_i \sum_{h=1}^{i} L_h + \Delta D_d L_i - \Delta L_j D_i$$

Again the first term is "normal", the second and third misc loses due to the absence of ΔD_j and ΔL_j in the summations.

Thus the full contribution to the "misclose" in area due to the approximate values assumed for line j is

$$- \Delta D_j\left[\sum_{h=1}^{j} L_h - L'_j\right] + \Delta L_j\left[\sum_{h=1}^{j} D_h - D'_j\right]$$

$$+ \Delta Dj \sum_{h=j+1}^{m} L_h - \Delta L_j \sum_{h=j+1}^{m} D_h$$

$$= - \Delta D_j\left[\sum_{h=1}^{j} L_h - \sum_{h=j}^{n} L_h\right] + \Delta L_j\left[\sum_{h=1}^{j} D_h - \sum_{h=j}^{n} D_h\right]$$

Similarly, the contributions of the k^{th} and m^{th} lines to a misclose in area are

$$- \Delta D_k\left[\sum_{h=1}^{k} L_h - \sum_{h=k}^{m} L_h\right] + \Delta L_k\left[\sum_{h=1}^{k} D_h - \sum_{h=k}^{m} D_h\right]$$

and

$$\Delta D_m\left[\sum_{h=1}^{m} L_h - \sum_{h=m}^{n} L_h\right] + \Delta L_m\left[\sum_{h=1}^{m} D_h - \sum_{h=m}^{n}\right]$$

Thus the third equation for computing three corrections is:

$$- \Delta D_j\left[\sum_{i}^{j} L_h - \sum_{j}^{n} L_h\right] + \Delta L_j\left[\sum_{i}^{j} D_h - \sum_{j}^{n} D_h\right]$$

be made manually before resort to the calculator, or automatically within a computer programme. Either way the equation for solution is

$$\underset{3\times 3}{\underline{A}}\ \underset{3\times 1}{\underline{X}} = \underset{3\times 1}{\underline{K}} \qquad (11)$$

Special Cases

Circumstances can arise leading to the determinant of A in (11) being zero, thus meaning that the solution is indeterminate. For example, if three distances are missing from lines which are parallel,

$$A = \begin{bmatrix} \text{Sin}\ \theta & \text{Sin}\ \theta & \text{Sin}\ \theta \\ \text{Cos}\ \theta & \text{Cos}\ \theta & \text{Cos}\ \theta \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and $|A| = 0$

because A can be obtained from a matrix

$$B = \begin{bmatrix} 0 & 0 & 0 \\ \text{Cos}\ \theta & \text{Cos}\ \theta & \text{Cos}\ \theta \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

by multiplying by a constant, Tan θ, each element of row 2 of B and adding it to the corresponding element of row 1.

There are five cases which will give rise to a zero determinant, although in two cases the problem is more apparent than real:

(i) Distances missing from 2 or 3 parallel lines in the same direction;
(ii) Bearings missing from 2 or 3 parallel lines in the same direction;
(iii) Distances missing from 2 parallel lines in the same direction and a bearing from a line perpendicular to them;
(iv) Bearings missing from the parallel lines in the same direction and the distance from a line perpendicular to them;
(v) Distance missing from one line and the bearing from another perpendicular to it.

It is immediately apparent that cases (iv) and (v) are non-problems, while (iii) reduces to (i) with two missing distances.

In these circumstances, where the matrix A is singular, a solution is possible only with additional information that will give rise to a fourth (and therefore redundant) equation. With a fourth equation it is then possible to delete one of the first two rows of A, that is, the rows containing sin θ and cos θ, because it is these that cause the zero determinant. In general, it is immaterial which row is eliminated, but when the direction is cardinal it must be that row for which sin θ or cos θ = 0.

For the case of missing parallel bearings, the fact that they are parallel is the additional information required to form the fourth equation, for example

$$2\,\Delta\theta j - \Delta\theta_k - \Delta\theta_m = 0,$$

For the case of missing lengths of parallel sides the additional information can be either of a desired ratio of the lengths of the sides, for example if $\iota_j = 2\iota_k$, and $\iota_k = 3\iota_m$, say, then provided the approximate values are in this ratio

$$\frac{1}{2}\Delta\iota_j - \frac{2}{3}\Delta\iota_k - \Delta\iota_m = 0; \text{ or}$$

their sum, in which case

$$\Delta\iota_j + \Delta\iota_k + \Delta\iota_m = \text{required sum} - \text{approx. sum} = k_4, \text{ say.}$$

Of the two, knowing the ratios is preferable because the sum equation can give rise to a zero determinant again, when the bearing is cardinal.

Conclusion

With the exceptions just mentioned, equation (11) offers a solution to all plane close problems involving the computation of missing bearing and distances.

Reference

CAPON, L. B. (1982) *Desk Top Calculators for Surveyors,* D.D.I.A.E. Toowoomba.

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