Computation of Constrained Vertical Curves

by Naguib F. Danial

Abstract. Fitting a parabolic curve of a given length to symmetrically connect two grade lines of given slopes is the normal case of vertical curves which is usually discussed in surveying books. Site conditions and/or existing structures generally impose some constraints on the design of vertical curves. Examples of these constraints are when the vertical curve is to pass through one, two, or three fixed points, or when its end points must have a definite difference in elevation. These cases require lengthy calculations or otherwise time-consuming trial and error solutions. Formulas have been derived to quickly solve these problems. Contrary to normal practice, the radius of curvature of the parabolic curve at its turning point is used instead of the commonly used rate of change of slope.

Introduction

Sight conditions and/or existing structures generally impose constraints on the design of vertical curves. Among these constraints are the cases when the vertical curve has to pass through one, two, or three fixed points. Formulas to solve these problems are included in Herman and Elzer (1970), where they are given separately for every possible location of these points. The positive y-axis in this publication was taken in the direction of the curvature, i.e., for summit curves opposite to that for sag curves. The station of highest or lowest point is taken sometimes with respect to the left tangent point and sometimes with respect to the right tangent point. For computer programing and practical calculations it is preferable to have formulas related to the same reference or coordinate system. The purpose of this work is, therefore, to derive such formulas which are valid to both summit and sag curves regardless of the location of the fixed points.

General

Vertical curves are used to effect gradual change between tangent grades. They can be circular, simple parabolic, or cubic parabolic. The simple parabola generally is the preferable curve in highway profile design and it closely approximates the circular curve. In Figure 1 one imagines a circular arc with radius k passing through the lowest or, generally speaking, the turning point T which is chosen as the origin of a U-V coordinate system. The U-axis is taken horizontally, i.e., coinciding with the tangent at the turning point. From the geometry of the circle

$$u^2 + (k - v)^2 = k^2 \qquad (1)$$

where u and v are the coordinates of any point P on the curve. Due to flat slopes the ordinate v is very small with respect to the radius k; v^2 can therefore be neglected. Equation (1) leads then to

$$u^2 = 2kv \qquad (2)$$

Equation (2) is an equation of a quadratic parabola with the radius k of the circular arc as a parameter. It can also take the form

$$v = u^2/2k \qquad (3)$$

Equations (2) and (3) are derived based on offset v being perpendicular to the U-axis or the tangent at point T. Because of assumed flat slopes, however, these equations are also valid for vertical offsets measured with respect to other inclined tangents. Accordingly, the vertical offsets P_1P_1' and TT′ in Figure 2 are equal. The first offset is measured with respect to the horizontal U-axis, while the second offset, also perpendicular to the U-axis, is related to the inclined tangent P_1T'. The relationship between any vertical offset q_i at a distance x_i from the tangent point P_1 is, therefore,

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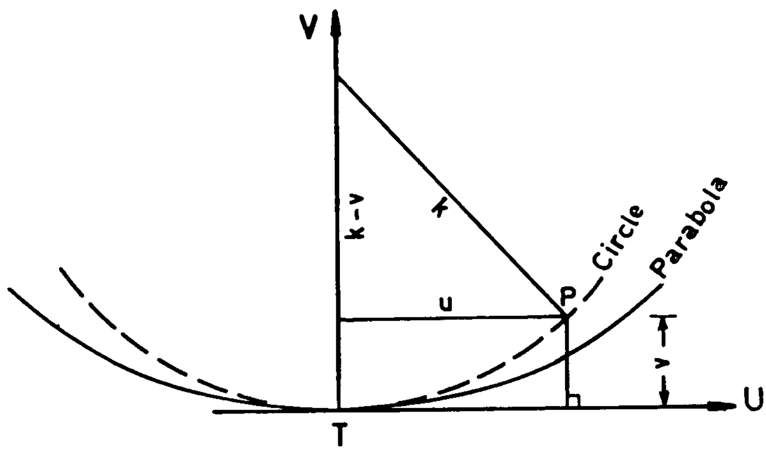

Figure 1. Parabola approximates circle when ordinate v is very small with respect to radius of curvature k.

$$q_i = x_i^2/2k \qquad (4)$$

In order to derive general formulas which are valid for both sag and summit curves, it is assumed that:

- The left tangent point is the point of beginning of curve,
- Gradients rising to the right are positive and gradients falling to the right are negative, and
- The parameter k is positive for sag and negative for summit curves.

The central angles α and β of the circular curve shown in Figure 3 are equal to the slope angles of grade lines AP_1 and P_2B respectively. If P_1 is the point of beginning and P_2 the end point of the curve, then

$$\tan\alpha = -u_1/(k - v_1) = -G_1/100 = -g_1 \qquad (5a)$$

$$\tan\beta = u_2/(k - v_2) = G_2/100 = g_2 \qquad (5b)$$

where u_1 and u_2 denote the horizontal distances between the turning point T and points P_1 and P_2, v_1 and v_2 are the ordinates of these points, G_1 and G_2 are the tangent grades expressed in decimal numbers. Since the ordinates v_1 and v_2 are very small with respect to k, they can be neglected. Equations (5a, b) will lead to

$$u_1 = kG_1/100 = kg_1 \qquad (6a)$$

$$u_2 = kG_2/100 = kg_2 \qquad (6b)$$

In the derivation of the following formulas only the decimal grades are used. Equations (6a, b) could also be obtained by differentiating the quadratic equation of the parabola (eq. (3)) with respect to the horizontal distance u and equating the derivative dv/du to the tangent slope g. By substituting the obtained value of u in equation (3), the vertical ordinate v_i at point P_i becomes

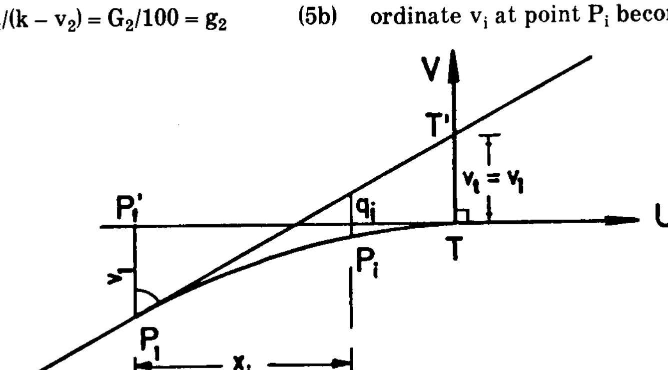

Figure 2. For same distances, vertical offsets between parabola and slope tangent are equal to those vertical offsets measured with respect to U-axis.

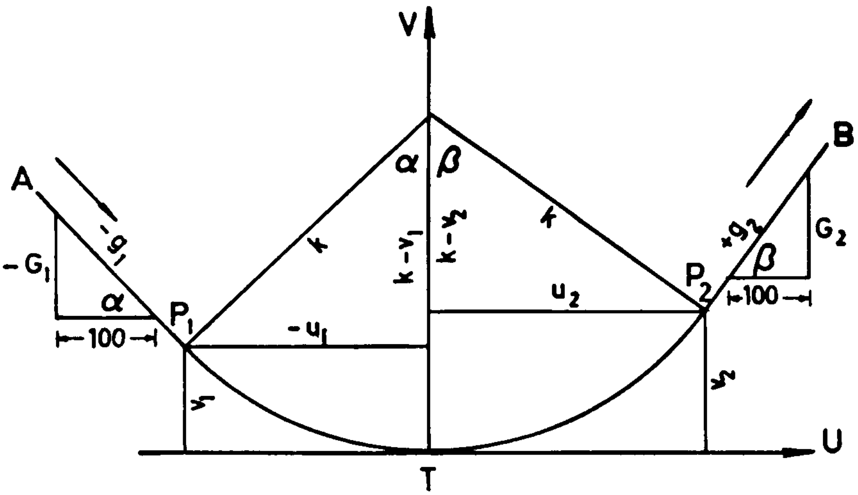

Figure 3. Relationship between slope angles, gradients, and radius of curvature.

$$v_i = u_i^2/2k = k^2g_i^2/2k = \tfrac{1}{2}kg_i^2 \quad (7)$$

The tangent grade g_i at a curve point P_i is, therefore,

$$g_i = u_i/k = \sqrt{2v_i/k} \quad (8)$$

Equation of Vertical Parabolic Curve

In practice another X-Y rectangular coordinate system is generally chosen with its vertical Y-axis passing through the point of beginning P_1. Its X-axis is the intersection of the plane of the vertical curve with the datum. The elevation y_i of curve point P_i above datum can be estimated from Figure 4 as follows:

$$y_i = y_t + v_i = y_1 - v_1 + v_i \quad (9)$$

where y_t and y_1 are the elevations above datum of the turning point T and the point of beginning P_1 respectively, and v_1 and v_i are the ordinates of points P_1 amd P_i in the U-V coordinate system. These ordinates are negative for summit curves and positive for sag curves. Considering equation (3) and not-

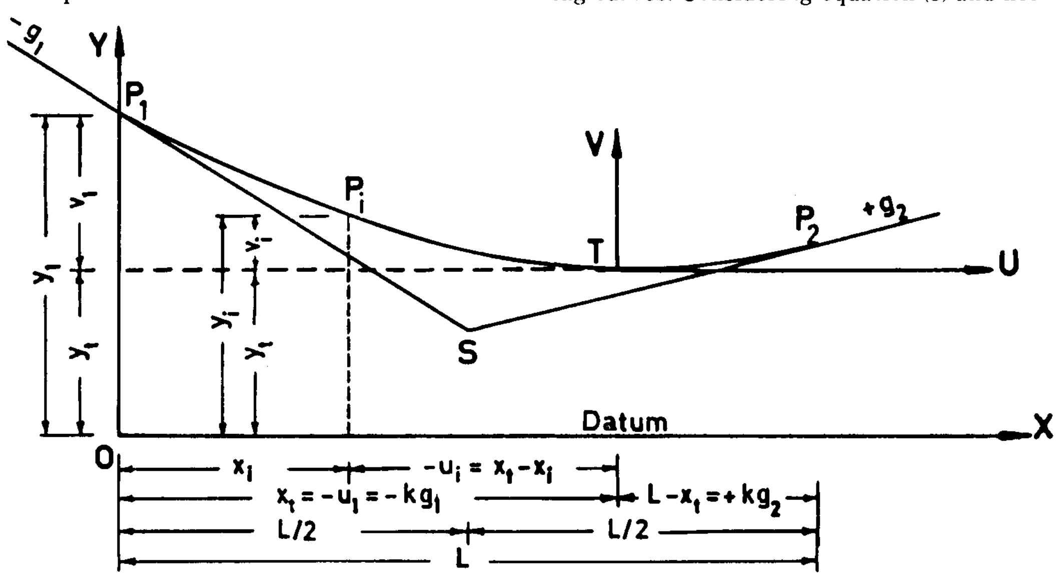

Figure 4. Relationship between U-V coordinate system with origin at turning point and the parallel X-Y coordinate system with Y-axis passing through point of beginning of curve P_1.

ing from Figure 4 that $u_i = u_1 + x_i$, equation (9) becomes

$$y_i = y_1 - v_1 + (u_1 + x_i)^2/2k$$
$$= y_1 - v_1 + u_1^2/2k + u_1x_i/k + x_i^2/2k \quad (10)$$

The second and third terms of the right side of equation (10) are numerically equivalent as expressed by equation (3). They are of different signs and therefore cancel. Substituting for u_1 by its value as obtained from equation (6a) and arranging terms leads to

$$y_i = (1/2k)x_i^2 + g_1x_i + y_1 \quad (11)$$

where y_i, x_i, y_1, and k are of the same linear units and g_1 is dimensionless. Equation (11) is exactly similar to the following commonly used equation for a vertical parabolic curve

$$y_i = (r/2)x_i^2 + g_1x_i + y_1 \quad (12)$$

where r is the rate of change of grade in percent per station, x_i is the distance in stations and y_i and y_1 are the elevations above datum in meters or feet. Equivalencing equation (11) with equation (12) rearranging and canceling yields

$$k = 1/r \quad (13)$$

i.e., the radius of curvature k at the turning point is the reciprocal of the rate of change of slope r.

In the design of vertical curves it is often required to determine the distance x_t and the elevation y_t of the turning point T. These quantities can be easily found from Figure 4 and equations (3) and (7).

$$x_t = -u_1 = -kg_1 \quad (14a)$$
$$y_t = y_1 - v_1 = y_1 - u_1^2/2k = y_1 - \tfrac{1}{2}kg_1^2 \quad (14b)$$

Relationship Between g_1, g_2, k, and Length of Curve L

The horizontal distance between the tangent points P_1 and P_2 is called the length L of the vertical curve. Knowing the radius of curvature k and the grades g_1 and g_2, one can determine this length from

$$L = -kg_1 + kg_2 = k(g_2-g_1) \quad (15)$$

which can be easily found from Figure 4. Equation (15) can also be written in the following forms:

$$g_1 = g_2 - L/k \quad (16)$$
$$g_2 = g_1 + L/k \quad (17)$$
$$k = L/(g_2 - g_1) \quad (18)$$

Curve Passing Through One Fixed Point

A common problem is finding the length of a vertical curve that will join two given tangents and will pass through a fixed point. The fixed point may be a road intersection or a point so determined to provide a minimum fill over a culvert or underground pipe or a minimum clearance below a bridge. Given values will usually be the gradients g_1 and g_2, the elevation y_a and station x_a of the fixed point A (Fig. 5), and the elevation and station of the intersection point S of the grade lines (called vertex). The following cases are considered:

Case 1.a: Beginning of Curve Known. The given elevation y_a of the fixed point A can be expressed mathematically as a function of the given data and the required length of the vertical curve by substituting equation (18) in equation (11).

$$y_a = (g_2 - g_1/2L)x_a^2 + g_1x_a + y_1 \quad (19)$$

Solving for the length of the curve L gives

$$L = ((g_2 - g_1)\cdot x_a^2)/(2(y_a - y_1 - g_1\cdot x_a)) \quad (20)$$

Example No. 1: Given the following data:

$G_1 = +2\%$, $G_2 = -3\%$, Station of A = 22 + 30, $y_a = 452.50$ ft., Station of $P_1 = 19 + 97$, and $y_1 = 451.18$ ft., compute the length L of the vertical curve which starts at P_1 and passes through point A.

Solution:

$$x_a = 2230 - 1997 = 233 \text{ ft.}$$

By equation (20)

$$L = \frac{(-0.03 - 0.02) \times 233^2}{2(452.50 - 451.18 - 0.02 \times 233)} = 406 \text{ ft.}$$

Case 1.b: Vertex of Curve Known. It is necessary in this case to compute the distance p and either offset q_1 or q_2 as shown in Figure 5. p is the horizontal distance between the fixed point and the vertex. q_1 and q_2 are the vertical distances, or offsets, between the fixed point and the first and second grade lines respectively. The distance p is computed as a function of the unknown length as follows:

$$p = x_a - L/2 \quad (21)$$

The offsets q_1 and q_2 can be found from

$$q_i = y_a - (y_s + g_ip) \quad (22)$$

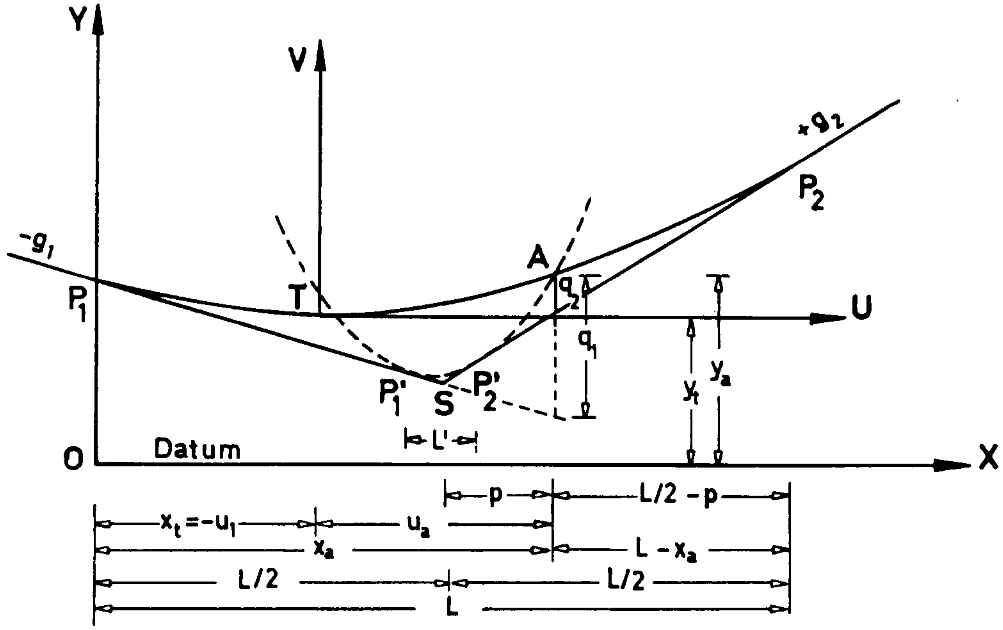

Figure 5. Two parabolas pass through a given fixed point A and touch the same two grade lines. The longer parabola is the sought curve.

where i is either 1 or 2. It should be noted that q_i will be negative for summit curves. By equation (4)

$$q_1 = (L/2 + p)^2/2k \quad (23a)$$

Similarly, considering the distance between P_2 and A (Fig. 5)

$$q_2 = (L/2 - p)^2/2k \quad (23b)$$

Rearranging equations (23a) and (23b), substituting for k by its value as obtained from equation (18), and collecting terms leads to the quadratic equations

$$L^2/4 + [p - ((2 \cdot q_1)/(g_2 - g_1))] \cdot L + p^2 = 0 \quad (24a)$$

$$L^2/4 - [p + ((2 \cdot q_2)/(g_2 - g_1))] \cdot L + p^2 = 0 \quad (24b)$$

Both equations are numerically the same. Solving either one leads to two values for L. The bigger one is the sought length of the vertical curve. The smaller value, L′ in Figure 5, is the length of the parabolic curve that also passes through point A but touches the grade lines near the vertex. It is apparent that this curve cannot be used.

Equations (24a, b) could have been obtained by substituting equations (21) and (22) in equation (19) and noticing that

$$y_1 = y_s - g_1 \cdot L/2 \quad (25)$$

The derivation shown here, however, is shorter and easier.

Example No. 2: A +2% grade line and a −3% grade line intersect at station 22+00 at elevation $y_s = 455.24$ ft. At station 22+30 the curve must pass through point A, of elevation $y_a = 452.50$ ft. Compute the length of the parabolic curve required to meet the given conditions and find its equation.

Solution:

By equations (21) and (22)

$p = 2230 - 2200 = +30$ ft.

$q_1 = 452.50 - (455.24 + 0.02 \times 30) = -3.34$ ft.

$q_2 = 452.50 - (455.24 - 0.03 \times 30) = -1.84$ ft.

Substituting either q_1 in equation (24a) or q_2 in equation (24b) gives

$$L^2 - 414.4\ L + 3600 = 0$$

Solving yields L = 405.5 ft. and L′ = 8.9 ft. The longer length is the required one as mentioned before. Substituting this length in equation (18) gives

$k = 405.5/(-0.03 - 0.02) = -8110$ ft. (summit curve)

By equation (25) the elevation of the point of beginning of the curve is

$$y_1 = 455.24 - 0.02 \cdot 405.5/2 = 451.185 \text{ ft.}$$

By equation (11) the equation of the curve is

$$y_i = -6.165 \cdot 10^{-5} x_i^2 + 0.02\, x_i + 451.185$$

from which the elevations of the intermediate stations can be determined. If the station and elevation of the highest point are required, then by equation (6a) the station is

$$x_t = -u_1 = -k \cdot g_1 = -(-8110) \cdot 0.02 = 162.20 \text{ ft.}$$

Substituting this value in the already found equation of the curve gives the elevation of the highest point as $y_t = 452.81$ ft. y_t can also be found directly from equation (14b)

$$y_t = y_1 - \tfrac{1}{2} k g_1^2 = 451.185 - \tfrac{1}{2}(-8110) \times (0.02)^2 = 452.81 \text{ ft.}$$

Case 1.c: Fixed Point—Highest or Lowest. In this practical case the elevations of the turning point and the vertex of the curve are known. Although the stationing of the turning point can be scaled approximately from the profile, it is assumed to be unknown and is not used in finding the length of the required curve. By equations (6a) and (18)

$$x_t = -u_1 = -kg_1 = -g_1 L/(g_2 - g_1) \tag{26}$$

By substitution in equation (11)

$$y_t = ((g_2 - g_1)/2L) \cdot (-g_1 L/(g_2 - g_1))^2 + g_1 \cdot (-g_1 L/(g_2 - g_1)) + y_s - g_1 \cdot L/2 \tag{27}$$

Reducing and solving for L leads to

$$L = (2(y_s - y_t)(g_2 - g_1))/g_1 g_2 \tag{28}$$

Case 1.d: Gradient g_1, Gradient g_a, and Offset q_1 at Fixed Point A Known. By substituting equation (18) in equation (3) one obtains

$$q_1 = L^2/2k = L(g_a - g_1)/2 \tag{29}$$

from which the length of the vertical curve is determined to be

$$L = 2q_1/(g_a - g_1) \tag{30}$$

Curve Passing Through Two Points

It is desired sometimes to design a vertical parabolic curve which passes through two points P_1 and P_2 at a distance L apart as shown in Figure 6. Depending on the data given, two cases arise. In the first case, the length of the curve L, the difference in elevation h between the two points, and the gradients at one of these points are given. In the second, the gradients at both points and either the length of the curve L or the difference in elevation h are known. The main concern in both cases is to find the radius of curvature k. This is best done on the U-V coordinate system with the turning point as the origin. The turning point can lie either inside or outside the distance P_1P_2.

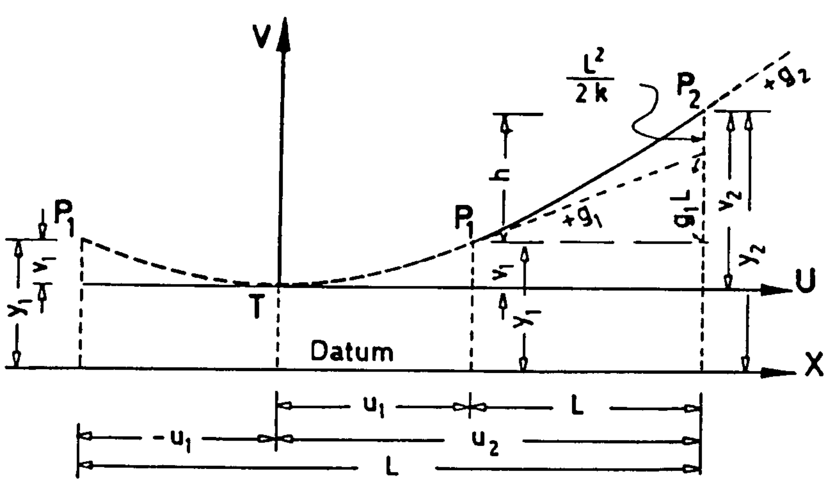

Figure 6. Parabolic curve passes through two points P_1 and P_2. Length L and difference in elevation h are given. Turning point can be inside or outside distance P_1P_2.

Case 2.a: L, h, g_1 (or g_2) Given; k, g_2 (or g_1) Required. From Figure 6 and equation (3)

$$h = y_2 - y_1 = v_2 - v_1 = u_2^2/2k - u_1^2/2k \quad (31)$$

Since $u_2 = u_1 + L$, we can substitute into equation (31) and rearrange to produce

$$h = ((u_1 + L)^2/2k) - u_1^2/2k = Lu_1/k + L^2/2k \quad (32)$$

Substituting for u_1 by its value as obtained from equation (6a) gives

$$h = g_1L + L^2/2k \quad (33)$$

From which the radius of curvature k can be found.

$$k = L^2/2(h - g_1L) \quad (34a)$$

The second gradient g_2 is determined afterwards by substituting the value of k in equation (17).

If g_2 is given instead of g_1, equation 34a will take the form

$$k = L^2/2(g_2L - h) \quad (34b)$$

g_1 can then be computed from equation (16).

It can be determined from Figure 6 that the offset between a tangent and any point on the curve at a distance L from the tangent point is $L^2/2k$ as determined previously and stated in equation (4).

Example No. 3: Points P_1 and P_2 are 70.00 m apart and have the elevations above datum of 20.51 m and 17.83 m respectively. If the grade at point P_1 is +2%, find the equation of the vertical parabolic curve that starts at P_1 and ends at point P_2. Compute also the station and elevation of the highest point T and the grade at point P_2.

Solution:

The difference in elevation between P_2 and P_1 is

$$h = y_2 - y_1 = 17.83 - 20.51 = -2.68 \text{ m}$$

By equation (34a)

$$k = 70.00^2/2(-2.68 - 0.02 \times 70.00) = -600.49 \text{ m}$$

According to equation (11), the equation of the parabolic curve is

$$y_i = -0.000833\, x_i^2 + 0.02\, x_i + 20.51$$

The distance to the highest point is obtained from equation (6a)

$$x_t = -u_1 = -k \cdot g_1 = -(-600.49) \times 0.02 = 12.01 \text{ m}$$

Substituting this distance in the already found equation of the curve gives the elevation of the highest point $y_t = 20.63$ m. By equation (17), the grade at point P_2 is

$$g_2 = 0.02 + 70.00/-600.49 = -0.097, \text{ or } G_2 = -9.7\%$$

Case 2.b: g_1, g_2, and L (or h) Given; k and h (or L) Required. Assume L is given and equate equation (18) with equation (34a)

$$L/(g_2 - g_1) = L/2(h - g_1L) \quad (35)$$

Reducing and arranging terms leads to the difference in elevation between the two tangent points as

$$h = ((g_1 + g_2) \cdot L)/2 \quad (36)$$

If h is given instead of L, L can be determined from equation (36)

$$L = 2h/(g_1 + g_2) \quad (37)$$

The radius of curvature k can now be determined using equation (18), (34a), or (34b). This is the normal case which is discussed in most surveying texts.

Example No. 4: Given $g_1 = +0.04$, $g_2 = +0.02$, and h = 2.40 m, compute L and k. Show how high the turning point of the curve is above point P_1.

Solution:

By equation (37),

$$L = 2 \times 2.40/(0.04 + 0.02) = 80.00 \text{ m}$$

By equation (18)

$$k = 80.00/(0.02 - 0.04) = -4000 \text{ m}.$$

The turning point is at a distance x_t from point P_1

$$x_t = -k \cdot g_1 = -(-4000) \times 0.04 = 160.00 \text{ m (outside curve } P_1P_2)$$

By equation (14) the difference in elevation between the turning point and point P_1 is

$$y_t - y_1 = -kg_1^2 = -\tfrac{1}{2}(-4000) \times 0.04^2 = +3.20 \text{ m}$$

the turning point being higher than P_1.

Curve Passing Through Three Points

It is possible to design a vertical parabolic curve to pass through three points provided

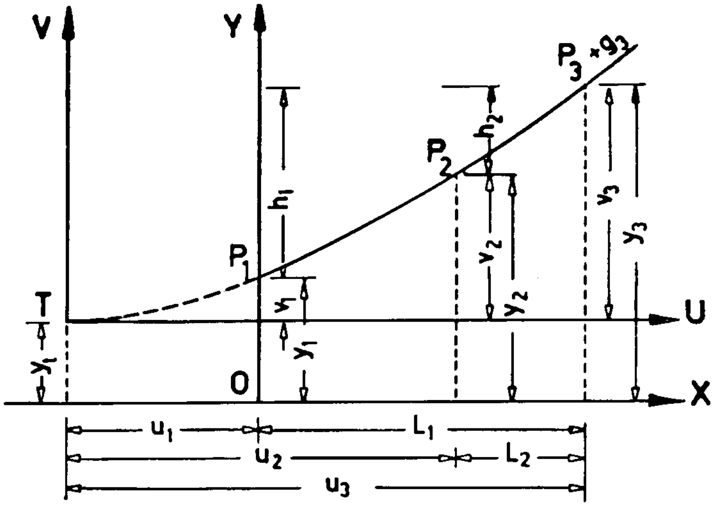

Figure 7. Parabolic curve passes through three points P_1, P_2, and P_3. Lengths L_1 and L_2 and differences in elevation h_1 and h_2 are given. Turning point can be inside or outside distance P_1P_3.

that the distances and the differences in elevation between them are known. Generally the stations and the elevations of the three points above datum are given from which the distances and the differences in elevation can be estimated. The solution is limited to finding the radius of curvature k. Referring to Figure 7

$$u_1 = u_3 - L_1 \quad (38)$$

and

$$u_2 = u_3 - L_2 \quad (39)$$

From equations (2) and (3)

$$h_1 = y_3 - y_1 = v_3 - v_1 = u_3^2/2k - u_1^2/2k$$
$$= u_3^2/2k - (u_3 - L_1)^2/2k = (2u_3L_1 - L_1^2)/2k \quad (40)$$

Solving for u_3 gives

$$u_3 = (2kh_1 + L_1^2)/2L_1 \quad (41)$$

Similarly, using equation (39)

$$h_2 = y_3 - y_2 = v_3 - v_2 = (2u_3L_2 - L_2^2)/2k \quad (42)$$

which leads to

$$u_3 = (2kh_2 + L_2^2)/2L_2 \quad (43)$$

Equating equation (41) with equation (43), then multiplying and collecting terms leads to

$$k = (L_1L_2(L_2 - L_1))/2(h_1L_2 - h_2L_1) \quad (44)$$

The location of the turning point can be found by substituting equation (41) in equation (38).

$$u_1 = (2kh_1 + L_1^2)/2L_1 - L_1 = (2kh_1 - L_1^2)/2L_1 \quad (45)$$

The elevation of the turning point is

$$y_t = y_1 - v_1 = y_1 - u_1^2/2k \quad (46)$$

The gradients g_1, g_2, and g_3 at points P_1, P_2, and P_3 respectively can be determined from equation (8).

Example No. 5: Find the equation of the vertical parabolic curve that passes through the three points P_1, P_2, and P_3 as shown in Figure 8. Compute also the station and elevation of the turning point and determine the grades at the given points. $L_1 = 105$ m, $L_2 = 15$ m, $y_1 = 22.17$ m, $y_2 = 22.45$ m, and $y_3 = 22.77$ m.

Solution:

$$h_1 = y_3 - y_1 = 22.77 - 22.17 = 0.60 \text{ m}$$
$$h_2 = y_3 - y_2 = 22.77 - 22.45 = 0.32 \text{ m}$$

By equation (44)

$$k = \frac{105 \times 15 \times (15 - 105)}{2 \times (0.60 \times 15 - 0.32 \times 105)} = +2881.1 \text{ m}$$

By equation (41)

$$u_3 = \frac{2 \times 2881.1 \times 0.60 + 105^2}{2 \times 105} = 68.96 \text{ m}$$

Equation (43) provides a check

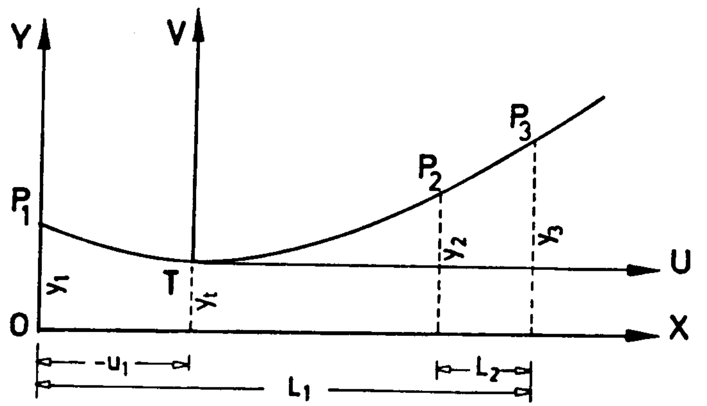

Figure 8. Curve passes through points P_1, P_2, and P_3 with turning point between points P_1 and P_2 (Numerical Example No. 5).

$$u_3 = \frac{2 \times 2881.1 \times 0.32 + 15^2}{2 \times 15} = 68.96 \text{ m}$$

$$u_1 = u_3 - L_1 = 68.96 - 105 = -36.04 \text{ m}$$

$$u_2 = u_3 - L_2 = 68.96 - 15 = 53.96 \text{ m}$$

The elevation of the turning point T can be determined from equation (46)

$$y_t = 22.17 - ((-36.04)^2/(2 \times 2881.1)) = 21.94 \text{ m}$$

The grades at points P_1, P_2, and P_3 are (by eq. (8))

$$g_1 = u_1/k = -36.04/2881.1 = -0.0125, \text{ or } G_1 = -1.25\%$$

$$g_2 = u_2/k = 53.96/2881.1 = +0.0187, \text{ or } G_2 = +1.87\%$$

$$g_3 = u_3/k = 68.96/2881.1 = +0.0239, \text{ or } G_3 = +2.39\%$$

Substituting k, g_1, and y_1 in equation (11) gives the required equation of the vertical curve.

$$y_i = 1.73544 \times 10^{-4} x_i^2 - 0.0125\, x_i + 22.17$$

Summary and Conclusion

Formulas to compute constrained vertical curves passing through one, two, or three fixed points are derived. An auxiliary U-V coordinate system with origin at the turning point and the U-axis parallel to datum was used for this purpose. The coordinates were then transferred to the X-Y coordinate system in which the X-axis lies in the plane of datum and the Y-axis represents elevation and passes through the first tangent point of the vertical curve. Several numerical examples are given to illustrate the use of the derived formulas.

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