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On Computations for Missing Elements of Closed Traverses

Dr. A. TÁRCZY-HORNOCH, Sopron, Hungary—In the March 1970 issue of SURVEYING AND MAPPING a very interesting paper was presented by Professor Root.

The importance of the problem described by Root is, in my opinion, the fact that it enables one to discover mistakes in the measurement by assuming the suspicious values of measuring as unknowns. From this point of view, cases have great importance when no real solutions can be found, because that hints at an error in the part supposed to be free of mistakes.

In an attempt to simplify the method, to limit the ambiguous solutions, and to give geometrical explanations to the values included, I offer my article for presentation in your *Journal* for discussion of precedents previously published there.

Remarks on Computations for Missing Elements of Closed Traverses

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IN his paper "Computations for Missing Elements of Closed Traverses," published in SURVEYING AND MAPPING (Vol. 30, No. 1, March 1970, pp. 91–93), James A. Root offers, among others, a solution of the case when the length S_y of one course of a closed traverse and the direction B_x of another course thereof are unknown. Direction B_y of the unknown S_y length and the length S_x pertaining to the unknown direction B_x are given. By means of symbols $\sin B_y = M$ and $\cos B_y = N$, as well as the values D and L required for the enclosure of the polygon, Root arrives at equations

$$S_x \sin B_x + S_y M + D = 0 \qquad (1)$$

$$S_x \cos B_x + S_y N + L = 0 \qquad (2)$$

where formula

$$S_y = \frac{-(MD+NL) \pm \sqrt{(MD+NL)^2 - (M^2+N^2)(D^2+L^2-S_x^2)}}{M^2+N^2}$$

is given for the calculation of S_y [Root's equation (8)]. There are two S_y solutions obtained therefrom, and Root seems to emphasize that a negative S_y has no sense here.

Although the formula given is correct, it can be further simplified, since

$$M^2 + N^2 = \sin^2 B_y + \cos^2 B_y = 1$$

and, therefore,

$$S_y = -(MD+NL) \pm \sqrt{(MD+NL)^2 - (D^2+L^2-S_x^2)}$$

is. The square root expression can also be further simplified:

$$\pm\sqrt{(MD+NL)^2 - (D^2+L^2-S_x^2)}$$
$$= \pm\sqrt{(M^2-1)D^2 + (N^2-1)L^2 + 2MNDL + S_x^2}$$
$$= \pm\sqrt{S_x^2 - N^2D^2 - M^2L^2 + 2MNDL}$$
$$= \pm\sqrt{S_x^2 - (ND-ML)^2}$$

which, in turn, gives

$$S_y = -(MD+NL) \pm \sqrt{S_x^2 - (ND-ML)^2}. \qquad (3)$$

It is easy to prove that the terms of equation (3) have simple geometric meaning. With some modifications, our Figure 1 corresponds to Root's Figure 2, and it has the courses of known direction and length properly fitted. R is not a traverse station but only a reference point. If, in the traverse, the directions involved are clockwise, in the case of Figure 1, the L and D values given by equations (1) and (2) are positive and negative, respectively. Since the given B_y is in quadrant II here, $\sin B_y = M$ will assume a positive, while $\cos B_y = N$ a negative value. It follows from Figure 1 that, in the given case,

$$-(MD+NL) = +S_{fR} \qquad (4)$$
$$(ND-ML) = +S_{Af} \qquad (5)$$

and, with the latter,

$$\sqrt{S_x^2-(ND-ML)^2} = |S_{1f}| = |S_{2f}|. \qquad (6)$$

Therefore, the two values of S_y will be as follows:

$$S_{y,1} = S_{fR} + |S_{1f}|$$
$$S_{y,2} = S_{fr} - |S_{1f}|$$

Their substitution into equations 1 and 2 will render the $\sin B_{x,1}$, $\sin B_{x,2}$, $\cos B_{x,1}$ and $\cos B_{x,2}$ values. Sine and cosine calculations represent not only a check-up but specify, jointly, the quadrant of the directions involved.

If

$$|S_{1f}| > |S_{fR}| \qquad (8)$$

then $S_{y,2}$ is of a negative value and has no meaning in the given case. Let us study now the geometric meaning of this inequality. According to Figure 1:

$$S_{1f}^2 = S_x^2 - S_{Af}^2$$

and

$$S_{fR}^2 = S_{Ar}^2 - S_{Af}^2$$

and, therefore, the inequality may be written as

$$S_x^2 - S_{Af}^2 > S_{AR}^2 - S_{Af}^2$$

where

$$S_x^2 > S_{Ar}^2 = D^2 + L^2 \text{ and } S_x > |\sqrt{D^2+L^2}|. \qquad (9)$$

In the case of inequality (9), i.e., if $S_x > |S_{AR}|$ only one positive S_y and, actually, one single solution at a given B_y is obtained. Since testing, which is the actually measured course of the two possibilities, is eliminated here, the longer S_x will be more favorable.

Figure 1 reveals, in addition, what a negative $S_{y,2}$ value means. In this case, point 2 corresponding to the direction of $B_{x,2}$ will not be at the length S_{R1} but at its extension below R, i.e., at the other side of R, whereby the direction in question towards R will be $B_y \pm 180°$ instead of B_y; in other words, it will *not* correspond to direction B_y as explained later on.

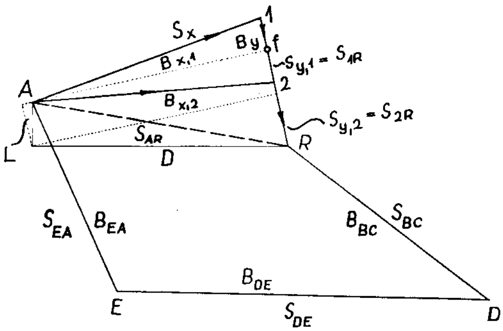

FIGURE 1.

It should be noted here that both S_y values according to equation (7) might be negative, if S_{fR} is negative, and its absolute value exceeds S_{1f}. Such a case is represented in Figure 2, where the traverse proper is omitted and only the values D, L and S_{AR} are indicated. Here not only D but also L is negative and, since this is in quadrant II, M is positive and N negative.

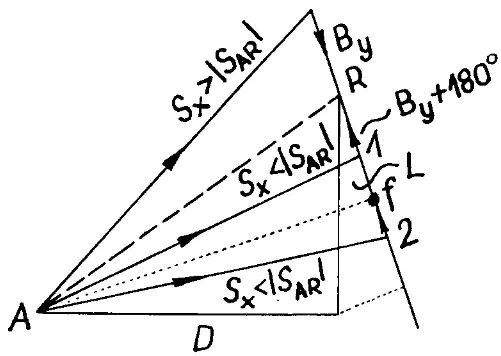

FIGURE 2.

Consequently $-MD$ is positive and $-NL$ negative, but because $|NL| > |MD|$, therefore $-(MD+NL) = S_{fR} < 0$. Since, furthermore, $|S_{1f}| < |S_{fR}|$, here—according to equation (7)—both S_y values are negative, which means that the clearance between A and R of the traverse is closed by the direction $B_y + 180°$ instead of B_y, and direction B_y does not lead to any solution. This is still more frequent with unchanged A and R, if B_y is in the third quadrant (Figure 3), because in this case—MD is negative too, and S_{fR} is given by the sum of the two negative terms and, according to equation (7), in case of an S_{1f} considerably exceeding the S_{1f} of Figure 2, both S_y values remain negative. There is no solution, either, when $S_x < |S_{Af}|$, because in this case the term under square root in equation (3) is negative, and extraction renders an imaginary value. So there is obviously no solution with either B_y or $B_y + 180°$. Both the two negative S_y values and the imaginary square root show that the traverse and direction values employed for the calculation of D and L, or the calculation proper must have contained errors.

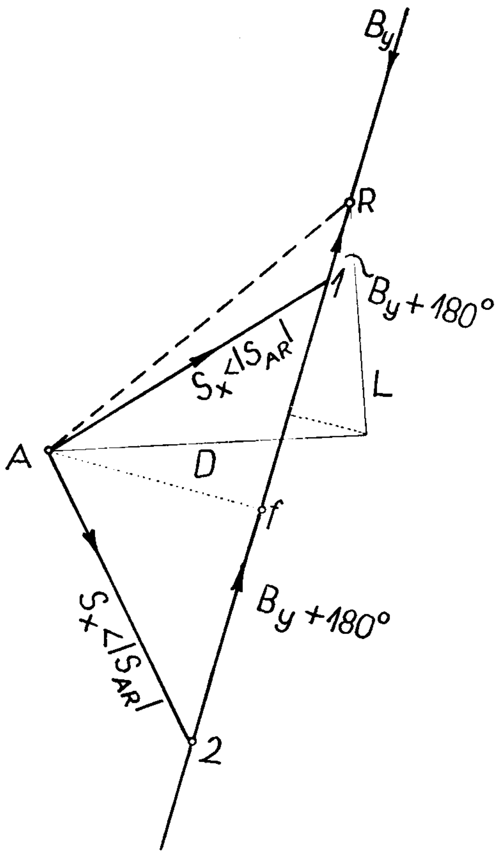

FIGURE 3.

If $B_y = B_{RA}$, then B_x may only be B_{AR}, and S_x as well as S_y must fall on line S_{AR}. In this case again only one solution is obtained, and $S_y = |S_{AR}| - S_x$. If $S_x > |S_{AR}|$, then there must be an error in either the measurement or the calculation.

The paper by Mr. Root deals also with the case where, in Figure 1, the two course lengths S_x, S_y are given, and their directions: B_x, B_y are sought for. He has introduced an auxiliary direction Θ expressed by

$$\sin\Theta = D/(D^2 + L^2)^{\frac{1}{2}}$$

and

$$\cos\Theta = L/(D^2 + L^2)^{\frac{1}{2}}, \qquad (10)$$

respectively, where, however, the direction proper is not easy to determine unequivocally, since $(D^2 + L^2)^{\frac{1}{2}}$ has a double sign and, even with an assumed positive extraction value, sine and cosine would render two reference values. It is, therefore, more reasonable to determine the tangent defining the direction quadrant:

$$\tan\Theta = \frac{D}{L} = \tan B_{RA} \qquad (11)$$

It is easy to realize that Θ means, in Figure 4, the direction from R to A. Let us indicate the angle at R of triangle ARa by $\triangle B_y$; since the three sides S_x, S_y and S_{AR} of the triangle are given

$$S_{AR} = |\sqrt{D^2 + L^2}|,$$

this angle can be expressed with the cosine theorem:

$$\cos\Delta B_y = \frac{S_y^2 + S_{AR}^2 - S_x^2}{2S_y \cdot S_{AR}} = \frac{-(S_x^2 - S_y^2 - S_{AR}^2)^2}{2S_y \cdot S_{AR}} \qquad (12)$$

And since $\cos \triangle B_y = \cos(360° - \Delta B_y)$, the $\triangle B_y$ at the other side of S_{AR} will also conform to equation (12) at R in triangle ARb, and thus two solutions will be obtained. In the given case

$$\text{a.) } B_y = B_{RA} + \Delta B_y + 180° \text{ and}$$
$$\text{b.) } B_y = B_{RA} - \Delta B_y + 180° \qquad (13)$$

If direction B_{AR} is taken instead of $B_{RA} + 180°$, then equation (13) will be simplified to

$$\text{a.) } B_y = B_{AR} + \Delta B_y$$
$$\text{b.) } B_y = B_{AR} - \Delta B_y \qquad (13a)$$

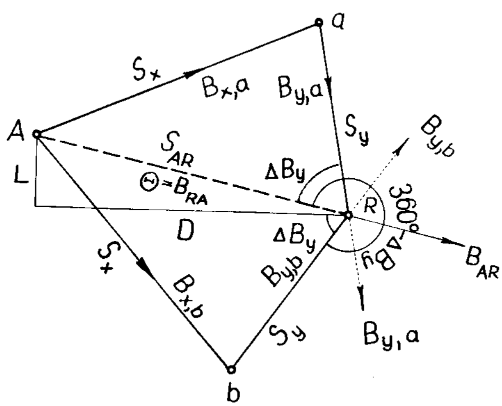

FIGURE 4.

If, in this case, $S_x + S_y > S_{AR}$, there are always two actually possible solutions existing which, as related to B_{RA}, are symmetrical. If

$S_x + S_y$ slightly exceeds S_{AR}, the definition will be all the more inaccurate, the smaller the difference is. If $S_x + S_y = S_{AR}$, then there is only one solution and S_x and S_y fall into course S_{AR}. If, finally, $S_x + S_y < S_{AR}$ then, from equation (12), there will be an absolute value greater than 1 obtained for cos ΔB_y, and no solution is obtained. This, in turn, will prove that there had been a mistake in either the data used for the calculation of the traverse, or in the calculation itself.

Our equation (12) is very similar to that published in Root's paper on page 93, in the right hand column. Furthermore, if ΔB_y is expressed from equation (13), taking into consideration that $B_{RA} = \Theta$ according to equation (11):

$$\Delta B_y = \Theta - B_y + 180°$$

and/or

$$\Delta B_y = B_y - \Theta - 180°$$

and the first ΔB_y is substituted into equation (12), then $-\cos(B_y - \Theta)$ is written instead of $\cos(B_y - \Theta - 180°)$ after multiplying both sides by -1, the equation published by Root on page 93 as referred to above will be obtained. If, on the other hand, the second ΔB_y value, i.e., $\Theta - B_y + 180°$ is substituted, the equation

$$\cos(\Theta - B_y) = (S_x^2 - S_y^2 - S_{AR}^2)/[2S_y\,(D^2 + L^2)^{\frac{1}{2}}] \qquad (14)$$

is obtained, which offers the second solution not even mentioned by Root. This follows, by the way, from

$$\sin\Theta \sin B_y + \cos\Theta \cos B_y = \cos(B_y - \Theta) = \cos(\Theta - B_y)$$

as well.

With ΔB_y calculated, angle aAR at A, and the corresponding symmetrical angle RAb at the other side of B_{AR}, then the similarly corresponding directions B_x can be calculated with the well-known method, or Root's equation (13) may be used for this purpose.

Derivation of equations 12 and 13 is not only simple but also defines the geometrical meaning of the values thus calculated.

The calculation technique presented by Root on page 93, as cited, has further disadvantages. As it is well-known, each cosine has two angle values associated. Equation (12) published by Root:

$$B_y = \cos^{-1}\{L/(D^2 + L^2)^{\frac{1}{2}}\} + \cos^{-1}\{(S_x^2 - S_{y,1}^2 - D^2 - L^2)/[2S_{y,1}\,(D^2 + L^2)^{\frac{1}{2}}]\}$$

will render, on the other hand, four ΔB_y values, even if $(D^2 + L^2)^{\frac{1}{2}}$ is always assumed to be positive. Equation

$$B_y = \cos^{-1}\{L/(D^2 + L^2)^{\frac{1}{2}}\} - \cos^{-1}\{(S_x^2 - S_{y,1}^2 - D^2 - L^2)/[2S_{y,1}\,(D^2 + L^2)^{\frac{1}{2}}]\}$$

obtained from (14) leads to solutions coinciding with the previous four results. The four ΔB_y values will, however, aggravate the selection of the two correct ones. For this reason, a solution according to our equations 12 and 13 is much more reasonable as it will render only the two correct values.

The problems discussed by Root may be of considerable importance in the detection of measuring and calculation errors of the traverse by introducing the doubtful data as unknown figures in the calculation, using the methods described above. Selecting the cases where, according to Figure 1, no actual solution is obtained for filling the gap between A and R, is all the more important as, in such cases, the errors in the $AEDR$ part assumed to be faultless and, therefore, the error must be sought here as pointed out above.

As a conclusion, it should be noted that the two problems discussed in this paper would have other solutions as well. Only the simplification and refinement of the two solutions presented by Root have been aimed at here.

Pages refer to James A. Root's article appearing in SURVEYING AND MAPPING, Vol. XXX, No. 1, March 1970.