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OPTIMIZATION TECHNIQUES IN CURVED AREA CALCULATION

Fouad A. Ahmed

Abstract

In this paper a technique for the determination of areas bounded by uneven curves is introduced. Two formulae are deduced for the numerical integration of the required area. The new formulae give high accuracy and surmount the drawbacks of the known methods.

Introduction

The problem of calculating areas bounded by curved lines is invariably encountered when surveying shores, ponds, hills and irregular properties; and also, when estimating the volume of earth cut and fill during excavation and embankment.

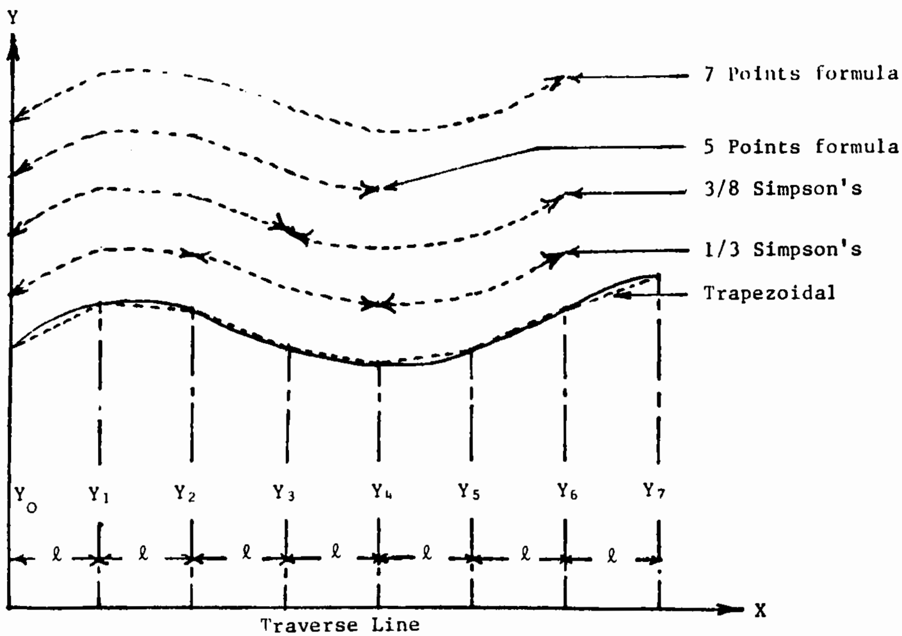

Fig. 1

Different techniques are used to obtain an estimation which is as close as possible to the unknown real value. The usual field work consists of dividing the traverse or the survey line into equal intercepts and then measuring the offsets to the curved boundary (Fig. 1). Accordingly, the total area is subdivided into a number of adjacent strips. The total area of these strips can then be calculated by different methods. If each strip is calculated separately the curvature of the boundary is ignored and the trapezoidal rule is applied. If the area of the strips is calculated in pairs every three offsets are assumed to be connected by a parabola and Simpson's rule is applied. The area by Simpson is given by one-third the width of a strip, multiplied by the sum of the two extreme offsets, four times the sum of the even offsets and twice the sum of the remaining odd offsets. The area of the strips can also be calculated in triplicate or more. More complicated formulae can be obtained such as; Four-point formulas (Simpson's three eights rule), Five-point formulas and Seven-point formulas. The application of such formulae is very limited, leaving Simpson's one-third rule the most popular in current survey work.

Drawbacks of Quadrature Formulae

In applying any of the previously mentioned quadrature formulae two main sources of errors are encountered. The first is the discontinuity of the fitted curves connecting the successive sets of offsets. Such discontinuity will create sharp junction points which do not match the actual curvilinear boundary. The misfit is appreciable if the boundary has curvature inflection points; and accordingly, the resulting error in the estimated area goes beyond an acceptable percentage. The second error source is introduced when the number of divisions is not equal to a whole number of intervals. In this case the solution is not unique and more than one answer can be obtained depending on the choice of the offsets corporated in the solution.

The New Approach

Consider the area bounded by the curvilinear (0)(n) the survey line "OX", the first offset y_0 and the last offset y_n, (Fig. 2). It is divided into equal intercepts of length "l". The measured offsets are $y_0, y_1 \dots ,y_n$. An odd number of strips, more than one, is chosen. In this mathematical model five strips are chosen and the general equation of the curvilinear (0)(5) may take this form:

$$Y = f(X) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4 + a_5 X^5 \qquad (1)$$

Considering the above conditions the factors a_0, a_1, ..., a_5 are found to be

$$a_o = y_o$$

$$a_1 = \frac{1}{120l} (-274y_0 + 600y_1 - 600y_2 + 400y_3 - 150y_4 + 24y_5)$$

$$a_2 = \frac{1}{120l^2} (225y_0 - 770y_1 + 1070y_2 - 780y_3 + 305y_4 + 50y_5)$$

$$a_3 = \frac{1}{120l^3} (-85y_0 + 355y_1 - 590y_2 + 490y_3 - 205y_4 + 35y_5)$$

$$a_4 = \frac{1}{120l^4} (15y_0 - 70y_1 + 130y_2 - 120y_3 + 55y_4 - 10y_5)$$

$$a_5 = \frac{1}{120l^5}(-y_0+5y_1-10y_2+10y_3-5y_4+y_5)$$

The area A_3 under the curvilinear (2)(3) is then obtained by integrating equation [1].

$$A_3 = \int_{2l}^{3l} f(X)dX,$$

and after some reduction,

$$A_3 = \frac{l}{1440}[11(y_0+y_5)-93(y_1+y_4)+802(y_2+y_3)]$$

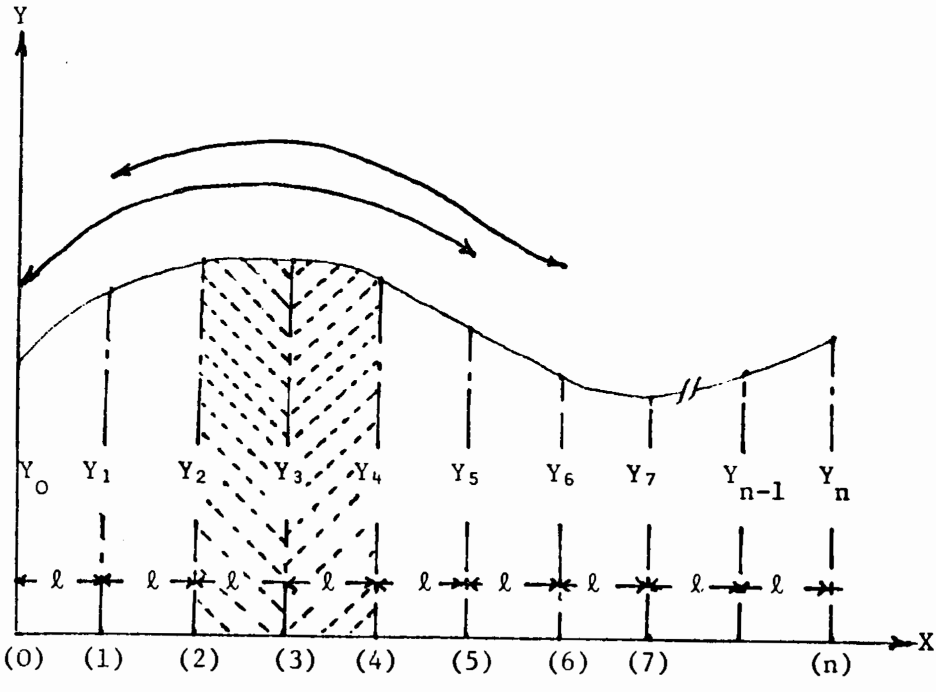

Fig. 2

To calculate the next area A_4 the section under the curvilinear (1)(6) is considered and with a similar procedure, the area of the strip A_4 is given by

$$A_4 = \frac{l}{1440}[11(y_1+y_6)-93(y_2+y_5)+802(y_3+y_4)]$$

It can be noted that the effect of discontinuity of the calculated successive areas is minimized.

For "n" number of strips the total area can be obtained by summation. The first and last two strips should be added. Their values are obtained from,

$$A_1 = \frac{l}{1440}(475y_0+1427y_1-798y_2+482y_3-173y_4+27y_5)$$

$$A_2 = \frac{l}{1440}(-27y_0+637y_1+1022y_2-258y_3+77y_4-11y_5)$$

$$A_n = \frac{l}{1440}(27y_{n-5}-173y_{n-4}+482y_{n-3}-798y_{n-2}+1427y_{n-1}+475y_n)$$

$$A_{n-1} = \frac{l}{1440}(-11y_{n-5}+77y_{n-4}-258y_{n-3}+1022y_{n-2}+637y_{n-1}-27y_n)$$

The final formula for the area A_f is

$$\begin{aligned} A_f = \frac{l}{1440} & (C_0+P_{5-n})(y_0+y_n)+(C_1+P_{6-n})(y_1+y_{n-1}) \\ & +(C_2+P_{7-n})(y_2+y_{n-2})+(C_3+P_{8-n})(y_3+y_{n-3}) \\ & +(C_4+P_{9-n})(y_4+y_{n-4})+(C_5+P_{10-n})(y_5+y_{n-5}) \\ & +k(y_6+y_7+\ldots+y_{n-6}) \end{aligned} \quad (2)$$

where

$P_0 = 16$, $P_1 = 107$, $P_2 = 306$, $P_i = 0$ (where $i < 0$)

$C_0 = 459$, $C_1 = 1982$, $C_2 = 944$, $C_3 = 1746$, $C_4 = 1333$, $C_5 = 1456$ and $K = 1440$.

More complicated equations can be obtained if sections consisting of more than five strips are considered for the calculation of the middle strip and vice versa. For further investigations sections consisting of three strips are examined. With the same procedure a similar, but simpler, formula is obtained. The area under the curved boundary in this case is given by:

$$\begin{aligned} A_a = \frac{l}{24} & a(y_0+y_n)+(b-P_n)(y_1+y_{n-1})+(C-P_{n-1})(y_2+y_{n-2}) \\ & +(C+1)(y_3+y_4\ldots+y_{n-3}) \end{aligned} \quad (3$$

where $a = 9$, $b = 28$, $C = 23$, $P_3 = 1$ and $P_d = 0$ (where $d \geqslant 4$).

Numerical Examples

The procedure to test the new method is to calculate the theoretical area under a certain function and compare it with the area estimated by different methods using the computed offsets. To illustrate the superiority of the new formulae, the following examples are given.

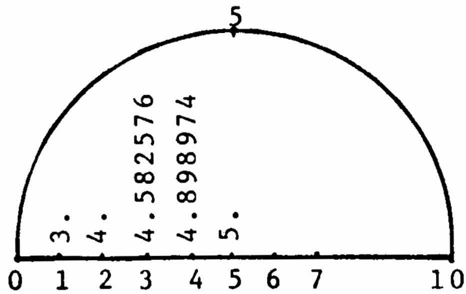

Fig. 3

In the first example the area of the half circle of radius 5, Fig. 3, is calculated and found to be $A_{exact} = 39{\cdot}2699$. Then it was divided into 10 strips with equal spacing and the corresponding offsets were calculated. Three methods are then applied to calculate the area by numerical integration; Simpson's A_s, the new A_f formula and the new A_a formula. The results are; $A_s = 38{\cdot}7522$, $A_f = 38{\cdot}6298$ and $A_a = 38{\cdot}7966$. Comparing A_f and A_a with the conventional A_s it is clear that the new formulae give better, but perhaps not a very significant improvement. This is because the curve under which the area is calculated has no points of curvature inflections.

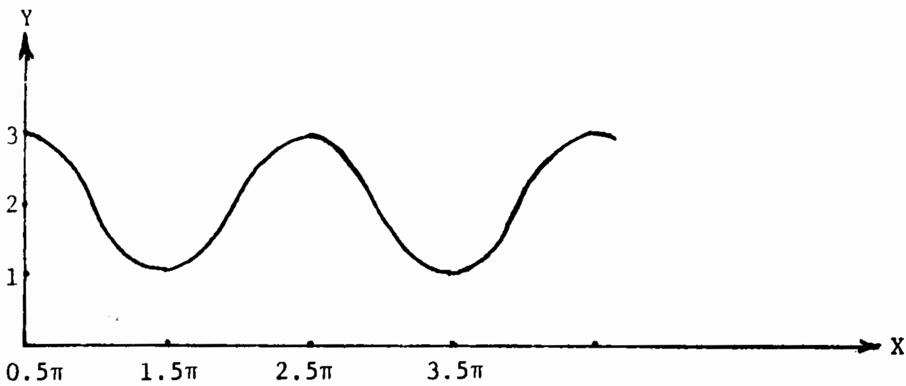

Fig. 4

In the second example the sin function is chosen since it resembles the configuration of natural curvilinears. The area under the sin curve is calculated after shifting the x-axis downward by two units. The ordinates of maximum and minimum points were then calculated and the area under the curve from $X = 0{\cdot}5\pi$ to $X = 16{\cdot}5\pi$ was calculated (Fig. 4). The exact area is obtained by integration and found to be 32π. Simpson's rule gave only 26·6667. Applying the new formulas A_f gave 31·3333 and A_a gave $29{\cdot}6\pi$. This shows that an average 15% reduction in error is reached when applying the new formulae.

Conclusion

The two new formulae give closer results to the estimated area under a curved boundary if equally spaced parallel offsets are given. Also the errors of discontinuity of the different sections in the conventional methods are minimized. And since the new solution is unique there is no restrictions for the number of offsets. The new formulae are recommended when calculating areas under curves that have several inflection points such as in lakes and in ponds, also, when calculating earth volume from levelling cross sections. The second formula is simpler and easy to apply since no complicated computations are required.

References

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