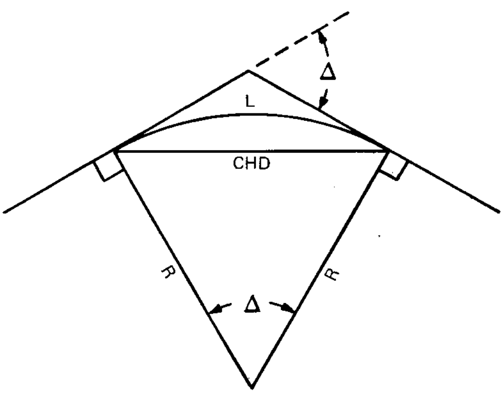
The Th Function

By WILLIAM C. THOMPSON
Ann Arbor, Michigan

THE following is a description of the *Th* Function, a new concept for finding data dealing with circular highway curves.

The *Th* Function (equation 1) is a correlation of arc length, chord, and central or delta angles of circular curves. It was primarily formulated to find circular curve data when only the arc length and chord are known. A secondary use evolved when it was found that a chord could be calculated in an extremely short period of time when the delta angle and arc length were known (equation 2), as opposed to the conventional method (equation 4) when the delta angle and radius were known.



Δ = central angle or delta angle
Δ_{RAD} = delta angle in radians
Δ_{Th} = *Th* functional value for the delta angle
L = length of arc
CHD = chord
R = radius
D = degree of curve

$$1)\ \Delta_{Th} = \frac{CHD}{L}$$

$$2)\ CHD = \Delta_{Th} L$$

$$3)\ L = \frac{CHD}{\Delta_{Th}}$$

$$4)\ CHD = 2R \text{ SIN} \frac{\Delta}{2}$$

$$5)\ L = R\,\Delta_{RAD}$$

$$6)\ D = \frac{5729.58}{R}$$

The *Th* Function was formulated when the author sought a unique equation to solve for the delta angle when only the arc length and chord were known. After many relentless hours of computation using algebra, trigonometry, analytic geometry, and calculus, it was found that a unique equation was unattainable. To find the unknown delta angle it was necessary to formulate an equation combining the two conventional equations used for finding the chord (equation 4) and length of arc (equation 5), and to calculate a constant for each delta angle.

The two equations were combined as follows:

Given:

Δ° = delta angle in degrees
L and CHD

$$4)\ CHD = 2R \text{ SIN} \frac{\Delta}{2}$$

$$5)\ L = R\,\Delta_{RAD}$$

Solution:

$$\frac{CHD}{L} = \frac{2R \text{ SIN}\frac{\Delta}{2}}{\Delta_{RAD}}.$$

Since:

$$\Delta_{RAD} = (\Delta^\circ)\left(\frac{\pi}{180^\circ}\right)$$

$$\frac{CHD}{L} = \frac{2 \text{ SIN}\frac{\Delta}{2}}{\Delta^\circ\left(\frac{\pi}{180^\circ}\right)}$$

$$7)\ \frac{CHD}{L} = \frac{\left(\frac{180°}{\pi}\right)\left(\text{SIN}\,\frac{\Delta}{2}\right)}{\left(\frac{\Delta°}{2}\right)} = \Delta_{Th}.$$

A calculator was then programmed for equation 7 and the quotient was found for every minute of arc from 0° to 180°. The results were combined into the copyrighted function book, *The Th Function for Every Minute of Arc From 0 to 180 Degrees.*

When curves for a plat or highway are designed, the plan usually gives the arc length, chord, delta angle, radius, and degree of curve. With the exception of the arc length and chord, given any two measurements, all curve data stated above can be calculated by using equations 4, 5, and 6. The *Th* Function has eliminated this exception.

The *Th* Function can be used in coping with erroneous curve data. For example, when using an old plat in which the correlation of the given delta angle, radius, arc length, and chord does not coincide, the chord and length of arc are found in error after computation using equations 4 and 5. It is more logical to believe that there is one error in either the delta angle of the radius than two errors in the length of arc and chord. By using the *Th* Function (equation 1), it will be found if the given delta angle coincides with the length of arc and chord. If the delta angle does coincide, one may assume the radius is in error. If the delta angle does not coincide, one may assume the delta angle is in error. Such errors should not occur, nevertheless they do. The previously marked lots and parcels of land have a considerable bearing on a survey; however the *Th* Function may help the surveyor.

The *Th* Function is extremely useful in the calculation of chords. Calculating a chord the conventional way by use of equation 4 uses the sine function, whereas calculation of a chord using the *Th* Function is a linear function. This functional difference may seem superficial before closer examination. Whenever a trigonometric function is used in a calculation, an accurate estimation of values is unattainable. A linear function can make use of proportions for such estimations. These estimations can reduce calculation time considerably, especially when a surveyor is in the field without a calculator. For example, with a given delta angle of 90° the *Th* Functional value is 0.900316316. Only four significant figures are needed for estimating 0.9003. By use of equation 2 the chord for a 100.00 ft. arc is then calculated.

$$CHD = (0.9003)\ (100.00\text{ ft.}) = 90.03\text{ ft.}$$

This can be done simply by moving the decimal point two places to the right. Thus, for a delta angle of 90°, the chord subtended by a 100.00 ft., arc is 90.03 ft. A proportion can be formed by using equation 1.

Given:

$$\Delta = 90°$$

$$1)\ \Delta_{Th} = \frac{CHD}{L}.$$

Solution:

$$0.9003 = \frac{90.03\text{ ft.}}{100.00\text{ ft.}}$$

$$0.9003 = \frac{CHD_1}{L_1}$$

$$\frac{90.03\text{ ft.}}{100.00\text{ ft.}} = \frac{CHD_1}{L_1}$$

$$\left(\frac{L_1}{100.00\text{ feet}}\right)(90.03\text{ ft.}) = CHD_1.$$

Hence, if

$$L_1 = 50.00\text{ ft.}$$

$$\frac{(50.00\text{ ft.})}{100.00\text{ ft.}}\ (90.03\text{ ft.}) = CHD$$

$$\tfrac{1}{2}\ (90.03\text{ ft.}) = 45.02\text{ ft.} = CHD_1.$$

In other words, for a given delta angle, the chord is proportional to the length of arc.

If accuracy is not crucial at the time of use, the third and fourth significant figures may be omitted, giving a ratio of chord to length of arc of 9:10. With this ratio the proportional corrections, i.e., the length of arc minus the chord to the length of arc (equation 8) is 1:10.

$$8)\ \frac{(L\text{-}CHD)}{L} = \frac{1}{10}$$

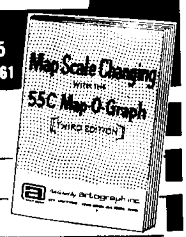
Thus, a 70.0 ft. length of arc has a corresponding chord of 70.0 ft. minus 7.0 ft. or 63.0 ft. This ratio may be used to give a rough distance to the location of a property marker.

It is important to note that this proportion can be used only with a given delta angle of 90°; however this is not a unique situation of the 90° delta angle. All delta angles have their unique proportion.

If more accuracy is needed, *The Th Function for Every Minute of Arc From 0 to 180 Degrees*, which has the *Th* Functional values calculated to nine significant figures, may be consulted. Computation is then followed by using equation 2 directly. The length of arc can be computed when the chord and delta angle are given by the use of equation 3.

The Th Function for Every Minute of Arc From 0 to 180 Degrees has not been published in quantity at this time, but the author's coworkers have found it to be helpful in the field.

In summary, the two main ideas presented were: the idea of the proportion of chord to arc length, and that curve data may now be computed when only the arc length and chord are known. The author believes that these two concepts may be useful to engineers and surveyors in the office and in the field. ■



Comment and Discussion

The pages of SURVEYING AND MAPPING are open to free and temperate discussion of all matters pertaining to the interests of the Congress. It is the purpose of this department to encourage comments on published material or the presentation of new ideas in an informal way.—EDITOR.

On 'The Th Function' by William C. Thompson

Published in SURVEYING AND MAPPING, Vol. XXXIV, No. 2, June 1974, pp. 151–153.*

ART COPELAND, *Anchorage, Alaska*—Although I am not a member of ACSM, I have been fortunate enough to be in the employ of one of your members who allows me to peruse your publications.

I have for many years searched (though not very diligently) for the curve equation which utilized the relationship of the arc length and long chord. You might well imagine my initial delight at finding "The *Th* Function"!

The article supplied an equation to be applied *directly* to the arc length and long chord. (A friend spent about five minutes programming his computer to spit out the attached [not shown here] *Th* tape).

I have encountered a few formulas that are generally attributed to Christaan Huygen. These are *approximate only*, but are generally applicable to "rough-guess" a curve. More recently a friend developed a "series-expansion" equation for me. This will be published in a book, distributed by Lyman Designs, Box 153, Weston, Mass. 02193.

Attached are appropriate excerpts from this book, which may be of interest to you.

The following are approximate formulas, generally attributed to Christaan Huygen (or Huyghen), and utilize the relationship between L and C:

$C' = \frac{3L + C}{8}$ For $\triangle = 120°$, error is about 1:400; for 180°, error is less than 1:80.

$C' = \frac{L + 3333C}{2.6667}$ Accurate to 1:1000 when M is less than $C/5$.

$M = \sqrt{\frac{(3C)\ (L - C)}{8}}$ Accurate to 1:10000 when M is less than $C/12$.

The following formulas were developed especially for this presentation through the efforts of Frank Frodsham, the local representative of Compucorp calculators in Anchorage, Alaska. These formulas, like those of Huygen, utilize the relationship between L and C.

The first formula (1) solves for $R\pm$, and the second formula (2) uses the solution of (1) to solve for R "exact." R may be further refined by resubstituting the result of (2) into a repeat of (2) and so forth, until there is no change within the desired number of significant figures of the desired end result. In other words, R "exact" is a misnomer, since it is an incommensurable figure in this series of equations. *For practical purposes, it is soluble*, by repetition of (2), to the desired degree of accuracy.

Terms:

$'$ = factorial

$5' = 5{\times}4{\times}3{\times}2{\times}1 = 120$

$7' = 7{\times}6{\times}5{\times}4{\times}3{\times}2{\times}1 = 5{,}040$

$9' = {\times}9{\times}8{\times}7{\times}6{\times}5{\times}4{\times}3{\times}2{\times}1 = 362{,}880$

$$R\pm = \sqrt{\frac{L^3}{24\ (L - C)}} \qquad (1)$$

$$R\text{ "exact"} = \sqrt{\frac{L^3}{24\left(L - C + \frac{L^5}{(2^4)\ (5')\ (R\pm^4)} - \frac{L^7}{(2^6)\ (7')\ (R\pm^6)} + \frac{L^9}{(2^8)\ (9')\ (R\pm^8)}\right)}} \qquad (2)$$

from formula (1)

Therefore, by substitution of 2^x and X':

$$R\text{ "exact"} = \sqrt{\frac{L^3}{24\left(L - C + \frac{L^5}{(16)\ (120)\ (R\pm^4)} - \frac{L^7}{(64)\ (5{,}040)\ (R\pm^6)} + \frac{L^9}{(256)\ (362{,}880)\ (R\pm^8)}\right)}} \qquad (2)$$

from formula (1)

* See also *Surveying and Mapping*, Vol. XXXIV, No. 3, Sept. 1974, p. 266

(continued p. 378)

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DISTINCTIVE RECENT MAPS

"Distinctive Recent Maps," compiled by Richard W. Stephenson of the Library of Congress, has been transferred to the ACSM *Bulletin*, effective with the May 1974 issue of that publication, and appears in the "Maps and Surveys" section. ■

The Surveyor and the Law cont'd

the neighbor's site in performing his duties. Again the neighbor called the municipal police and they in turn intercepted the survey crew as they were leaving. The third surveyor was taken into police custody and the neighbor swore out a warrant for his arrest. He was taken to police headquarters and an arraignment date was set.

At this point, the firm engaged its attorney to represent the accused. The attorney and accused appeared at the arraignment and a trial date was set.

During the trial the attorney was able to show that neither the complainant nor the police had actually told each surveyor to leave the premises and not to trespass. The judge, therefore, dismissed the case.

The lost time to the surveyor, the aggravation and embarrassment of being arrested, the employer's lost man-hours and the firm's cost of the attorney's fees might, in the future, be prevented by making surveyors aware of the consequences of trespassing. —from *The Michigan Surveyor*, Vol. 9, No. 2, 1974. ■

Comment and Discussion cont'd

For further refinement of R "exact"

$$R\text{ "exact"} \sqrt{\frac{L^3}{24\left(L - C + \dfrac{L^5}{(1920)\ (R^4)} - \dfrac{L^7}{(322{,}560)\ (R^6)} + \dfrac{L^9}{(92{,}897{,}280)\ (R^8)}\right)}} \qquad (2)$$

from formula (2)

$$R\text{ "exact"} \sqrt{\frac{L^3}{24\left(L - C + \dfrac{L^5}{(1920)\ (R^4)} - \dfrac{L^7}{(322{,}560)\ (R^6)} + \dfrac{L^9}{(92{,}897{,}280)\ (R^8)}\right)}} \qquad (2b)$$

from formula (2a)

And so forth, until there is no change within the desired number of significant figures of the end result. ■

dary locations are chaotic and seldom tied to a precise control system, most engineering surveys are made for a single purpose use, practically no underground works are tied to a common uniform control system, and information systems and data banks are not being coordinated between agencies. Some of the urgent needs of the 21st Century are: (1) The development and implementation of a modern land register; (2) A conversion of all title boundaries to a coordinate system; (3) The development and maintenance of a reliable information system and surveying data banks; and, (4) Conversion of all records to the metric system . . . .

Since the new surveyor must be knowledgeable in route, construction, land, topographic, and geodetic surveys, as well as in boundary control and legal principles, photogrammetry and geodesy, it is not reasonable to assume that the knowledge necessary can be gained through a civil engineering program, an apprentice program, or other miscellaneous routes. It is also not reasonable to assume that the average present practitioner can research, develop, and implement the new systems required . . . .

These are some of the basic reasons for the actions taken by the ACSM Board in March 1974. We must also keep in mind the time span factor of practitioners. Most newly qualified professionals will be involved in practice for a period of about 40 years. Many of the people who qualify in the early 1980s will still be in practice in the 21st century. If the United States is to remain a major power in the world and maintain a viable society during this period of time, the needs set forth above, in addition to many others, must be attained. In my mind, for our profession to settle for a lesser goal or objective than that adopted by the Board would be naive and fatuous.

There are a few misconceptions or misunderstandings in Mr. Bryan's letter that require clarification.

At no time during these considerations has there been any intent on the part of ACSM to exclude from registration any individual who successfully follows the work-study method of advancement. In the proposed revisions to the Model Registration Law, now being considered by the National Council of Engineering Examiners, there is still a provision for the individual without a college degree to obtain registration as a land surveyor through meeting the work experience requirements and passing the written examinations. It should be noted, however, that the uniform written examinations are being gradually upgraded so that by 1979 the subject material covered will parallel the academic subjects taught in a four-year baccalaureate program. Because of this, the number of individuals who obtain registration in the future without a degree will probably be significantly reduced.

Mr. Bryan seems to believe that an individual with a four-year degree in surveying could become registered a year after graduation. This is not correct. Such an individual would be permitted to take the fundamentals examination upon graduation but would be required to obtain a minimum of four years of additional experience and then pass the principles and practice examination before becoming registered.

There is one other, though important, misunderstanding Mr. Bryan seems to have which I wish to correct. **The major demand for the educational requirement has come, not from educators, but from professionals in private practice such as myself.**

On 'The Th Function' by William C. Thompson

Published in SURVEYING AND MAPPING, Vol. XXXIV, No. 2, June 1974, pp. 151–153*

ELBERT F. BASSHAM, Urban Engineering, Corpus Christi, Texas—The massive formulas for the radius of a curve given only the arc length and chord described in "Comment and Discussion," published in *Surveying and Mapping*, Vol. XXXIV, No. 4, December 1974, p. 363, produced one very definite feeling—that there must be an easier way. After a period of thought, some derivation, and some refinement I produced an iteration much simpler to use and hereby named "The Bassham Iteration."

Derivation is as follows:

Let R = radius, Δ = central angle in radians, L = arc length, C = chord length, then from the circle, $L = \Delta R$ and $C = 2R \sin(\Delta/2)$.

Solving for R, $R = L/\Delta$ and $R = \dfrac{C}{2\sin(\Delta/2)}$.

Combining, $L/\Delta = \dfrac{C}{2\sin(\Delta/2)}$.

Simplifying, $\Delta = \dfrac{2L}{C}\sin(\Delta/2)$.

Let $K = C/L$ and $\theta = \Delta/2$, then $\theta = \dfrac{1}{K}\sin\theta$.

Now, setting $f(\theta)=\theta-\frac{1}{K}\sin\theta$ and applying Newton's method of iteration which says that for $x_2=x_1-f(x_1)/f'(x_1)$ (f' is the derivative of f) x_2 is nearer a root of $f(x)$ than x_1, provided x_1 is a reasonable estimate. To obtain an even better estimate, substitute x_2 for x_1 obtaining a new x_2. This iteration may be repeated until sufficient accuracy is achieved. Applying this to $f(\theta)=\theta-\frac{1}{K}\sin\theta$ the derivative $f'(\theta)=1-\frac{1}{K}\cos\theta$.

$$\text{So, } \theta-\frac{f(\theta)}{f'(\theta)}=\theta-\frac{\theta-\frac{1}{K}\sin\theta}{1-\frac{1}{K}\cos\theta}=\frac{\frac{1}{K}\sin\theta-\frac{1}{K}\theta\cos\theta}{1-\frac{1}{K}\cos\theta}=\frac{\tan\theta-\theta}{K\sec\theta-1}.$$

So, for

$$\theta_2=\frac{\tan\theta_1-\theta_1}{K\sec\theta_1-1}\quad\text{(Bassham Iteration)}$$

$$K=C/L,\quad \theta=\Delta/2\quad\text{(in Radians)}.$$

θ_2 is nearer to the $\Delta/2$ for the curve than θ_1. Thus Δ can be computed by a few iterations of the above.

To illustrate the simplicity of its use, I submit an HP-35 program.

ENTRY	OPERATION	ENTRY	OPERATION	ENTRY	OPERATION
C	↑	π	x		÷
L	÷	180	÷	1	−
	↑		−		*RCL*
	↑		*RCL*		x⇄y
θ					
(in degrees)	*STO**		x⇄y		÷
	RCL		*STO*	180	x
	tan		*R*↓	π	÷ (yields next θ)
	RCL		cos	Go to *	

Simply repeat the loop in this program until two successive values of θ are within the accuracy desired.

Example: for $L=523.60$ and $C=517.64$ if we choose $\theta_1=20$ degrees

$$\left.\begin{array}{l}\theta_2=16.40192\\ \theta_3=15.15995\\ \theta_4=14.99912\\ \theta_5=14.99912\end{array}\right\}\ \pm5\times10^{-5}$$

which yields $\Delta=29°59'54''$ and $R=1000.06$.

* *See Surveying and Mapping*, Vol. XXXIV, No. 3, Sept. 1974, p. 266, and No. 4, Dec. 1974, p. 363.

On 'Adjustment of a Quadrilateral—A New Look at an Old Problem' by Professor Herbert W. Stoughton

Published in **Proceedings of the American Congress on Surveying and Mapping, Fall Convention, Sept. 10–13, 1974**

Author Stoughton requests that the following changes be made in Appendix A of his paper:

Corrigenda

PAGE 256

Formula (A–3) should read:

$$\varepsilon=1.27615*10^{-9}\,[1.-0.006768658\sin^2\phi]^2K$$

Formula (A–4) should read:

$$\varepsilon=\alpha\, a_1\, b_1\sin C_1$$

PAGE 257

Formula (A–5) should read:

$$\varepsilon=\alpha*10^{-9}\,[a_1\, b_1\sin C_1]$$

Comment and Discussion

The pages of SURVEYING AND MAPPING are open to free and temperate discussion of all matters pertaining to the interests of the Congress. It is the purpose of this department to encourage comments on published material or the presentation of new ideas in an informal way.—EDITOR

On 'Trespass . . . A Recent Case' [The Surveyor and the Law]

Published in SURVEYING AND MAPPING, Vol. XXXIV, No. 4, Dec. 1974, p. 362

F. HENRY SIPE, LLS, Elkins, W. Va.—The crew chief could perhaps have avoided the dire consequences if he first had visited the adjoiner's residence, informed the occupant of his mission, and requested information about the location of the adjoiner's markers, etc. Tactics, ethics, and courtesy indicate this to be a safer route. If such a visit did not find a friendly neighbor, it would have served to alert the crew to an unfriendly atmosphere that pointed to a need for extra precautions.

On 'Statistical Tests as Guidelines in Analyses of Adjustment of Control Nets' by Urho A. Uotila

Published in SURVEYING AND MAPPING, Vol. XXXV, No. 1, Mar. 1975, pp. 47–52

Author Uotila requests that the following changes be made in his paper: page 52, left column, the inequality formula should read

$$\frac{\hat{V}'P\hat{V}_2 - \hat{V}'P\hat{V}_o}{\hat{V}'P\hat{V}_o} \cdot \frac{DF}{b} > F_{,b,n-u,a}$$

On 'Adjustment of a Quadrilateral—A New Look at an Old Problem' by Herbert W. Stoughton

Published in PROCEEDINGS OF THE AMERICAN CONGRESS ON SURVEYING AND MAPPING, Fall Convention, Sept. 10–13, 1974

Author Stoughton requests that the following changes be made to his paper:

Corrigenda

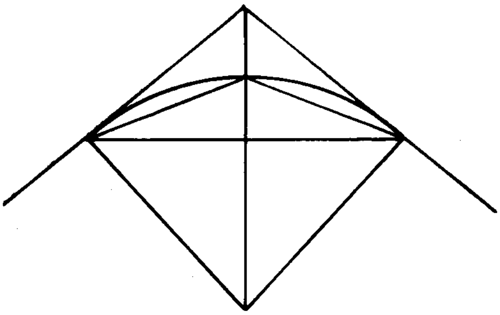
Page 259—Formula (B–10) For: +log sin *(8)*; should read: −log sin *(8)*.
Page 274—loading No. 4 and No. 5: For (CONTINUE).
Page 278—instructions read (↑ Key).

On 'The Th Function' by William C. Thompson

Published in SURVEYING AND MAPPING, Vol. XXXIV, No. 2, June 1974, pp. 151–153*

JOHN E. COMBS, L.A. County Road Dept., Los Angeles, Calif.—Mr. Thompson states that after many relentless hours of computation using algebra, trigonometry, analytic geometry, and calculus he was unable to find a simple method of calculating a delta and radius when only the long chord and length of arc are known. *My solution follows:*

* See also *Surveying and Mapping*, Vol. XXXIV, No. 3, Sept. 1974, p. 266, and No. 4, Dec. 1974, p. 363.



$LC = 49.98$
$ARC = 50.00$

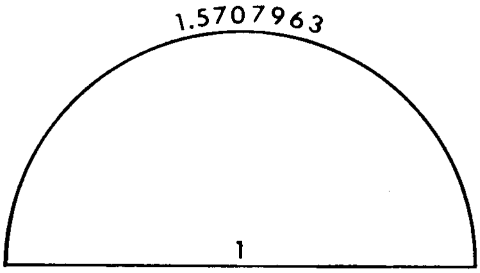
$$\frac{3\ ARC + LC}{8} = sub\ chord\ \frac{199.98}{8} = 24.9975$$

$$\frac{\frac{1}{2} LC}{sub\ chord} = \cos\frac{\Delta}{4}\ \frac{24.99}{24.9975} = .9996999$$

$\cos .9996999 = 1.403815°$
$1.403815 \times 4 = 5.61526°$
$5.61526°$ in $RAD = .098005$

$$R = \frac{ARC}{RAD} = \frac{50}{.098005} = 510.18$$

This answer is correct; but if the delta is large an error will enter because the formula 3 $ARC +$ LC divided by 8 is only an approximation. This error can be readily calculated and eliminated. The maximum possible error would be approximately 1% with a delta of 180 degrees and would be inversely proportional to the square of the delta for any angle less then 180 degrees. Example:



$DELTA = 180°$
$R = .5$

$$ARC = \frac{\pi}{2} = 1.5707963$$

$LC = 1$

$$\frac{3\ ARC + LC}{8} = .714086\ sub\ chord$$

$$\frac{\frac{LC}{2}}{sub\ chord} = \cos\ \frac{\Delta}{4}$$

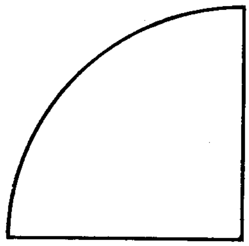
$$\frac{.5}{.714086} = .7002324$$

$\cos .7002324 = 45.55433$
$45.55433 \times 4 = 182.21732°$
$182.21732° = 3.180291\ RAD$

$$\frac{ARC}{RAD} = .4939159$$

$$\frac{.4939159}{.50} = .0098783\%\ \text{error}$$

The following error is calculated from a delta of 90 degrees, in order to show the inversely proportional characteristic.



$DELTA = 90°$ $LC = .707107$
$R = .5$

$$ARC = \frac{\pi}{4} = .7853981$$

Calulated radius would be .501863

$$\frac{.501863}{.50} = .003726\%\ \text{error}$$

Thus, since the error may be calculated for any delta, it may be readily eliminated.

Example: If a delta were calculated using my method and the delta or the radius were so large that the error were of concern then one would merely insert knowns into my formula using the previously calculated delta and immediately find the percent of error and add or subtract them from his own calculated radius.

The *Th* function was a nice invention and I hope Mr. Thompson made a lot of money selling his tables, but this method is quick and accurate and free. Just give me all the glory!

A. Gerald Wiatrowski, Waukegan, Ill. (Ill. RLS 1997)—Although I am not presently a member of ACSM, I do look forward to reading your publications which are passed on to me by Richard L. Thacker, a Fellow member.

The "Bassham Iteration" (*Surveying and Mapping*, Vol. XXXV, No. 1, Mar. 1975, pp. 70–71) is a very interesting way to solve a curve problem, given only the arc length and chord. If the solution could be derived in such an easy manner on the HP–35, it should be even easier on the HP–65.

The following program is one which I have developed for use on the HP–65. The solution is derived by trial and error solutions of the two basic formulas involved:

$$(1)\ L = \frac{(\Delta)\,(\pi)\,(2)\,(R)}{360°}$$

(2) $C = 2R\left(\sin\frac{\Delta}{2}\right)$

where R = radius, Δ = central angle in degrees-minutes-seconds, L = arc length, and C = chord length. By combining equations (1) and (2) and simplifying, the following equality results:

(3) $\frac{(\Delta)(\pi)}{(360°)(L)} = \frac{\sin\frac{\Delta}{2}}{C}$

The HP–65 program automatically assumes a $\Delta/2$ angle of 181 degrees and reiterates similar to the "Bassham Iteration" until the equality in equation (3) is reached.

Although this program takes about 30 seconds to run, it should be of benefit to HP–65 owners who are faced with solving curve problems given only the arc length and chord.

Ed. Note: As a registered land surveyor of Illinois, it's time to join or rejoin your national professional organization—ACSM.

Program

1. LBL	16. TAN	31. π	46. 2	61. STO 2	76. RCL 2
2. A	17. RCL 6	32. ÷	47. gR↓	62. 3	77. R/S
3. STO 3	18. g	33. RCL 3	48. STO 6	63. 6	78. RCL 6
4. RTN	19. π	34. RCL 5	49. GTO	64. 0	79. f
5. LBL	20. X	35. ÷	50. B	65. RCL 5	80. →D.MS
6. A	21. 1	36. RCL 6	51. LBL	66. X	81. R/S
7. STO 5	22. 8	37. f	52. 2	67. 4	82. RCL 3
8. 1	23. 0	38. COS	53. +	68. ÷	83. R/S
9. 8	24. ÷	39. ÷	54. 2	69. g	84. RCL 1
10. 1	25. –	40. 1	55. ÷	70. π	85. RCL 6
11. STO 6	26. 1	41. –	56. STO 6	71. ÷	86. f
12. RTN	27. 8	42. ÷	57. 2	72. RCL 6	87. TAN
13. LBL	28. 0	43. RCL 6	58. X	73. ÷	88. X
14. B	29. X	44. gx≤y	59. f	74. STO 1	89. R/S
15. f	30. g	45. GTO	60. →D.MS	75. R/S	90. RCL 5
					91. RTN

Operation

Instructions	*Input Data/Units*	*Keys*	*Output Data/Units*
1. Enter program			
2.	C	A	
3.	L	A	
4. Curve Solution		B	R
5.		R/S	Δ (Deg-min-sec)
6.		R/S	Δ/2 (Deg-min-sec)
7.		R/S	C
8.		R/S	T
9.		R/S	L

NORMAN G. DENNIS, G.E. Co., Technical and Support Svs. Dept., Bay St. Louis, Mississippi—Dennis sent copies of pp. 96–97 of *Signals, Systems and Communications*, by B. P. Lathi, assoc. professor of Electrical Engineering, Bradley University, published by John Wiley & Sons, Inc., New York. These pages say that, "This function plays an important role in communication theory and is known as the *sampling function*, abbreviated by *SA(x)* for convenience." The formula then shown:

$$Sa(x) = \frac{\sin x}{x}$$

Dennis goes on to say that, "The *Sa*( ) notation was invented some time after 1952," adding that he doesn't know exactly when. He also says that the book mentioned above is "a beautiful clear book."

On 'The First Method To Adjust a Traverse Based on Statistical Considerations' by Herbert W. Stoughton

Published in SURVEYING AND MAPPING, Vol. XXXIV, No. 2, June 1974, pp. 145–149

IRA H. ALEXANDER, *Los Angeles, Calif.*—In the June 1974 issue of *Surveying and Mapping*, Herbert W. Stoughton, Assistant Professor of Civil Technology at the State University of New York, Alfred, New

Comment and Discussion

The pages of SURVEYING AND MAPPING are open to free and temperate discussion of all matters pertaining to the interests of the Congress. It is the purpose of this department to encourage comments on published material or the presentation of new ideas in an informal way.—EDITOR

On 'The Th Function' by William C. Thompson

Published in "Surveying and Mapping," Vol. XXXIV, No. 2, June 1974, pp. 151–153

RESPONSE TO A COMMENT

WILLIAM C. THOMPSON, author of article—On John E. Combs' comment in "Comment and Discussion," published in *Surveying and Mapping*, Vol. XXXV, No. 2, June 1975, pp. 172–173 . . . .

I was glad to see that someone finally solved the arc length-chord problem when I began reading Mr. Combs' comment on the Th Function. However, I believe I found that it had not been solved but only approximated as I neared the end of the article. My examination goes as follows.

Mr. Combs writes that the error ". . . would be inversely proportional to the square of the delta for any angle less than 180 degrees." This indicates:

Where k = proportionality constant

$$\frac{\text{“}R_{cal}\text{”}}{R} = k\,\frac{1}{\Delta^2}$$

Actually, the error $(R\text{-}R_{cal})$ is approximately proportional to the central angle squared. In his examples, the calculated radius for delta equals 180° is too small, while the calculated radius for delta equals 90° is too large. I believe I found an arithmetic error in the 90° radius. My calculations show the radius to be 0.498170. With this I attempted to find k as follows.

The method reduces to equation 910 with the elimination of degree-radian conversion. I use only radian angular measurement in the following for practicality:

Let: L = length of arc, CHD = chord length, Δ = central angle, R = radius, R_{cal} = radius calculated by this method, Error = $R - R_{cal}$, % Error = $(R_{cal} - R)\ 100/R$, $E = R_{cal}/R$

The method equates to:

$$R_{cal} = \frac{L}{(4)\cos^{-1}\left[\dfrac{(4)\ CHD}{3L + CHD}\right]} \quad \ldots\ldots (1)$$

For: $\Delta = 45° = \pi/4$ radians, $R = 0.5000\ldots$, $CHD = 0.382683433$, $L = 0.392699082$ --- $R_{cal} = 0.499524114$, $\text{Error}_{45} = 0.000475886$, % Error = −0.0952, $E = 0.999048228$,

$$k_1 = \text{Error}_{45}/^2 = 0.000771477$$
$$k_2 = {}^2(E) \quad = 0.616263$$

For: $\Delta = 90° = \pi/2$ radians, $R = 0.5000\ldots$, $CHD = 0.707106782$, $L = 0.785398164$ --- $R_{cal} = 0.498169782$, $\text{Error}_{90} = 0.001830218$, % Error = −0.366%, $E = 0.9963396$,

$$k_3 = \text{Error}_{90}/^2 = 0.000741759$$
$$k_4 = {}^2(E) \quad = 2.458369$$

For: $\Delta = 180° = \pi$ radians, $R = 0.5000\ldots$, $CHD = 1.000\ldots$, $L = 1.57079633$, --- $R_{cal} = 0.493915560$, $\text{Error}_{180} = 0.00608444$, % Error = −1.22%, $E = 0.98783112$,

$$k_5 = \text{Error}_{180}/^2 = 0.000616483$$
$$k_6 = {}^2(E) \quad = 9.7495023$$

Large errors occur as Δ approaches 0°.

$k_1 \neq k_3 \neq k_5$; $k_2 \neq k_4 \neq k_6$: hence, the error is not inversely proportional or proportional to the square of the delta.

Do the k values vary enough to make a difference?

Let $R' = R_{cal}$ + correction for difference in R

$R'_{45} = 0.499524 + k_3\ (\pi/4)^2 = 0.499982$

. . . 0.00369% Error

$R'_{180} = 0.493916 + k_3\ (\pi)^2 = 0.501236$

. . . 0.247% Error

A common survey accuracy of 1:10000 is equal to 0.010%. This was not reached in one on the above calculated radius values. This method will work quite nicely, with negligible error, on small delta angles not approaching 0°. Since there is an unpredictable error in all delta angles the Th function has yet to be solved without the use of iterations, progressions, and computers.

In his last paragraph, Mr. Combs says that he hopes the author "made a lot of money selling his tables." The tables were written and copyrighted in 1971; only a few copies have been published. Many facets of the Th Function are not needed since the onset of calculators with field mobility.

In his last sentence, Mr. Combs writes, "Just give me all the glory." Let us not discredit the men who have written to *Surveying and Mapping* with good methods for finding the Th Function in the past one and a half years. ■

Comment and Discussion

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National Vertical Control Net

HERBERT W. STOUGHTON, P.E., L.S.–On Wednesday, 16 May 1973, the following article appeared in the *Federal Register* [vol. 38, No. 94, page 12840] on the change of the official name of the vertical control network employed in the United States. The text follows:

NATIONAL VERTICAL CONTROL NET

Proposed Action

May 7, 1973.

Elevations of marked points (benchmarks) in the National Vertical Control Net are based on the "Sea Level Datum of 1929." Since this datum was derived from the overall average sea level of 26 tide stations, the official elevation at any particular one of these tide stations does not necessarily reflect the actual local "mean sea level." In order to avoid such apparent confusion and the costly errors that may result through failure to consider local sea level when engineering projects are undertaken, it is proposed to change the present name of the vertical control datum from the "Sea Level Datum of 1929" to the "National Geodetic Vertical Datum of 1929."

This change is proposed to be effective on or before July 2, 1973. Comments on this proposed action may be directed to the Director, National Ocean Survey, NOAA, Rockville, Md. 20852.

Robert M. White
Administrator

[FR Doc. 73-9694 Filed 5-15-73; 8:45 am]

Re: Spelling—'Meter' or 'Metre'

ACSM PUBLICATIONS COMMITTEE–Since there are controversial opinions as to the spelling of meter/metre and inasmuch as some feel very strongly as to what the spelling should be, the Publications Committee has reached a decision that until a specific spelling of the word has been accepted by ACSM, the spelling used by the author will be used throughout the article, or the abbreviation "m." will be used when applicable for the sake of conformity in all cases.

On 'The Th Function' by William C. Thompson

Published in 'Surveying and Mapping,' Vol XXXIV, No. 2, June 1974, pp. 151–153*

ELBERT BASSHAM, Corpus Christi, Texas–I would like to protest the deceiving presentation by A. Gerald Wiatrowski (SURVEYING AND MAPPING, Vol. XXXV, No. 2, June 1975, pp. 173–174).

He claims to have written an HP65 program to "reiterate" the equation

$$\frac{(\Delta)(\pi)}{(360°(L))} = \frac{\sin\frac{\Delta}{2}}{C},$$

but on inspection of his program listing I find that this is indeed not the case (see steps 14–50). Actually he is using the iteration

$$\theta = \frac{\tan\theta - \theta}{K\sec\theta - 1}, \text{ where } \theta = \frac{\Delta}{2} \text{ and } K = \frac{C}{L}$$

which is the "Bassham Iteration" (SURVEYING AND MAPPING, Vol. XXXV, No. 1, March 1975, pp. 70–71).

In addition I would like to make three suggestions for the program.

*See also SURVEYING AND MAPPING, Vol. XXXIV, No. 3, Sept. 1974, p. 266; No. 4, Dec. 1974, p. 363, 378; Vol. XXXV, No. 1, March 1975, pp. 70–71; No. 2, June 1975, pp. 172–173; No. 3, Sept. 1975, p. 263.

1. Assume a $\Delta/2$ angle of 90 degrees (Step 8-10)

or

2. Allow the operator to select a Δ angle (surveyors should be able to do this within 10 degrees)

and

3. Rework the logical jump (Step 44) to check for equality within 10^{-4} instead of the full capacity of the calculator, (10^{-10}).

These changes will produce an accuracy of ±0.5 sec. and should operate much more quickly.

For those who might be interested, here are two additional variations of the "Bassham Iteration":

Version 2: Given the arc length (L) and chord (C) of a curve, the iteration

$$R = \frac{1}{2}\,\frac{L - C\sec\theta}{\theta - \tan\theta}\left(\text{where } \theta = \frac{L}{2R}\right)$$

will yield the radius of the curve. (Note: θ is in radians)

Version 3: Given the arc length (L) and the tangent (T) of a curve, the iteration

$$R = \frac{\tfrac{1}{2} L\sec^2\theta - T}{\theta\sec^2\theta - \tan\theta} \text{ where } \theta = \frac{L}{2R}$$

will yield the radius of the curve. (Note: θ is in radians).

I have HP35 programs for these iterations and would be pleased to send a copy to anyone requesting them. (Elbert Bassham, Urban Engineering, P.O. Box 6355, Corpus Christi, Texas 78401)

WILLIAM P. LOVELAND, R.S., P.E., Instructor Land Surveying, Bethlehem Area Vocational Technical High School; Project Engr., Penn. Power and Light Co.—I read with interest William C. Thompson's Article "The *Th* Function" in the June 1974 Quarterly Journal. I also read with interest J. E. Combs' article in the June 1975 Quarterly Journal. My geometry book also gives the approximate rule for finding the length of arc of a segment as "From eight times the chord of half the arc, subtract the chord of the whole arc and divide the remainder by 3." To this point I agree with Mr. Combs. However, for $\cos^{-1}$ .9996999 I get 1°24'13" and this I convert to 1.403611° and 4 times this is 5.614444, which is .0979905 radian. This in turn gives a radius of 510.25', not the 510.18' given in Mr. Combs' article. The following is the method I use in a problem of this kind.

Checking find:

The length of the chord of half the arc is

$$\frac{199.98}{8} = 24.9975 \text{ as before}$$

$m = 24.9975 \sin 1°24'13" = 24.9975 \times .0244952 = .612318'$

$R^2 = 24.99^2 + (R - .612318)^2$

$R^2 = 624.5001 + R^2 - 1.224637R + .374934$

$R = \frac{624.87503}{1.224637} = 510.25'$ which does check

also $2 \times 510.25 \sin 2°48'26" = 49.98'$ which checks finally $\frac{2\pi\Delta}{360} = .0979905$ radian

and $L = .0979905 \times 510.25 = 50.00'$ and this also checks.

In addition let's look at Mr. Thompson's problem where the arc is 100' and the chord is 90.03'. By Mr. Combs' method:

$$\frac{3ARC + LC}{8} = \frac{390.03}{8} = 48.75375$$

$\cos\frac{\Delta}{4}\,\frac{45.015}{48.75375} = .9233135 = \cos 22°35'05"$

$= \cos 22.5833472°$

$22.5833472° \times 4 = 90.333388° = 1.5766157$ radians

and $R = \frac{100}{1.5766157} = 63.426997$

$2 \times 63.426997 \times \sin \tfrac{1}{2}\Delta = 89.96'$ instead of 90.03'

However, if the problem is worked algebraically as above:

$m = 48.75375 \sin 22°35'05" = 18.723833'$

$R^2 = 45.015^2 (R - 18.723833)^2$

$R = \frac{2026.3502 + 350.58192}{37.447666} = 63.473437$

and $2 \times 63.473437 \times .70919486 = 90.03'$ which agrees with the given information, also

$\frac{100}{63.473437} = 1.5754621$ radians

$= 90.26729°$

and $\frac{2\pi\Delta}{360} \times 63.473437 = 100.00'$ which also agrees.

It seems, therefore, that if the problem is solved algebraically the errors are negligible. I can see where Mr. Thompson's book can be a valuable tool in practice, but, as a teacher, I feel a student should understand the solution to a problem and be able to solve that problem without tables. In practice, however, I agree with Mr. Thompson, as time is money, and after my students have mastered a problem I will then show them the shortcuts.

RALPH GIDDINGS, P.E., Chairman, Technology Div., Coastal Carolina Community College—The series of comments inspired by Mr. Thompson's "Th Function" has been most interesting, but so far all contributors have stopped short of generalizing the solution to cover all possible simple curve variations. There are seven essential elements in a simple curve. These elements and their symbols are:

1. The delta angle, Δ (if measured in degrees), or δ (if measured in radians).
2. The radius, R.
3. The tangent, T.
4. The external, E.
5. The middle ordinate, M.
6. The long chord, C.
7. The arc length, L.

Given any two of these, the other five may be determined. But how many surveyors have tried to solve all 21 possible combinations?

The primary curve elements are Δ and R. When these two elements are known, the others can be calculated directly, since:

$$\text{Secondary elements} = f(R, \Delta) \qquad (1)$$

The particular forms of equation (1) used for calculating the various secondary elements are the basic simple curve formulas:

$$T = R \tan \Delta/2 \qquad (1.1)$$

$$E = R\left[\frac{1}{\cos \Delta/2} - 1\right] \qquad (1.2)$$

$$M = R(1 - \cos \Delta/2) \qquad (1.3)$$

$$C = 2R \sin \Delta/2 \qquad (1.4)$$

and

$$L = R\delta \qquad (1.5)$$

These equations avoid the use of the external secant in (1.2) and the versed sine in (1.3) because those functions are not directly available on hand-held electronic calculators, such as the HP series. The relationship

$$\delta = \Delta \frac{\pi}{180} \qquad (2)$$

is also avoided because many such calculators can be operated in radian mode. If not, equation (2) will be required.

Note that, when the delta angle appears in an equation as "Δ," it is measured in degrees; when it appears as "δ," it is measured in radians.

When both primary curve elements (Δ and R) are known, the secondary elements all follow directly from the basic simple curve equations.

When one primary curve element (either Δ or R) plus any secondary curve element are known, the missing primary element (R or Δ) can be found by a rearrangement of the applicable basic curve equation. The rest of the solution is then as before.

The real problem arises when the primary elements are not known, but instead any two secondary elements are. In this event we have two equations (the two applicable basic curve equations) and two unknowns (Δ and R). Since these equations are consistent and independent, a solution will exist.

There is a method of attack that will always work under this condition. It consists of selecting the simpler of the two equations and solving it for R. This expression for R is then substituted into the other equation leaving one equation with one unknown, the delta angle. This procedure, which, with humility in keeping with the rest of this series, I have called *The Giddings Method*, leads to seven cases. These cases are discussed in turn below.

Case I. If T and C are known, equation (1.1) is solved for R and this expression is substituted into equation (1.4). The use of trigonometric identities and simplification then leads to:

$$\text{Cos } \Delta/2 = C/(2T) \qquad (3)$$

Knowing Δ, R follows directly, and the solution reverts to the use of the basic curve formulas.

Case II. If E and M are known, equation (1.3) can be solved for R and this expression substituted into equation (1.2). Mathematical manipulation then leads to:

$$\text{Cos } \Delta/2 = M/E \qquad (4)$$

The solution then proceeds as before.

Case III. If T and E are known, equation (1.1) can be solved for R and this expression substituted into equation (1.2). Manipulation then leads to:

$$\text{Cos } \Delta/2 = \frac{1 - (E/T)^2}{1 + (E/T)^2} \qquad (5)$$

The solution preceeds as before.

Case IV. If M and C are known, equation (1.3) can be solved for R and this expression substituted into equation (1.4). Mathematical manipulation on this leads to:

$$\text{Cos } \Delta/2 = \frac{(C/2M)^2 - 1}{(C/2M)^2 + 1} \qquad (6)$$

Again, the solution proceeds as before.

Case V. If E and C are known, equation (1.4) can be solved for R and this expression for R substituted into equation (1.2). This produces:

$$E = \left[\frac{\frac{1}{2}C}{\sin \Delta/2}\right]\left[\frac{1}{\cos \Delta/2} - 1\right] \qquad (7)$$

which may be reduced to:

$$p \cos^3 \Delta/2 + p \cos^2 \Delta/2 + \cos \Delta/2 - 1 = 0 \qquad (7.1)$$

where

$$p = (2E/C)^2$$

Of course, equation (7.1) could be solved, but with an electronic hand-held calculator it is easier to use a trial-and-error method than to fight through a cubic. The most convenient elements to know are the primary elements, so The Giddings Method simply assumes a value for Δ. This assumed value of Δ is substituted into equation (7) and the value of E corresponding to that Δ is calculated. If this result agrees with the known value of E, the assumed value of Δ was correct. If it does not, another value of Δ is chosen and the iteration is repeated until Δ has been determined to any desired degree of accuracy. Once Δ is known, R follows directly and we proceed as before.

Case VI. If T and M are known, equation (1.1) can be solved for R and this result substituted into equation (1.3). The result is:

$$M = \frac{T}{\tan \Delta/2}(1 - \cos/2) \qquad (8)$$

which may be reduced to:

$$\cos^3 \Delta/2 - \cos^2 \Delta/2 + q \cos \Delta/2 + q = 0 \qquad (8.1)$$

where

$$q = (M/T)^2$$

As in Case V, a trial-and-error solution of equation (8) for Δ using a capable hand-held calculator is easier than a solution of the cubic equation. The rest of the solution is as before.

Case VII. When L and any other secondary element are known, a new problem arises. When equation (1.5) is solved for R, we get:

$$R = L/\delta \qquad (9)$$

and, substituting this expression for R into equations (1.1) through (1.4) in turn, we get:

$$T = (L/\delta)(\tan \delta/2) \qquad (10)$$

$$E = (L/\delta)\left[\frac{1}{\cos \delta/2} - 1\right] \qquad (11)$$

$$M = (L/\delta)(1 - \cos \delta/2) \qquad (12)$$

$$C = 2(L/\delta)(\sin \delta/2) \qquad (13)$$

These equations present the difficulty that stimulated this series. They all contain both an algebraic term (δ itself) and a transcendental term (a trigonometric function of δ). Therefore, they cannot be solved in closed form, but require either tables (like the "Th Function"), an iterative process (like the "Bassham Iteration"), the use of infinite series, etc.

The tables required do not presently exist. (Note that the "Th Function" used to solve equation (13) is only one of four such tables that would be required for a complete solution.) Further, the creation of such tables in an age of the hand-held electronic calculator would be an anachronism. The "Bassham Iteration" is an effective algorithm for the solution of equation (13) (three more are required), but these algorithms would present problems either of memory or of library search if they must be looked up. Infinite series, particularly in the case of equations (11) and (12), might get pretty "messy."

The Giddings Method again assumes a trial value of the delta angle (this time in radians). This value of δ is substituted into the proper equation, (10) through (13), and the corresponding value of the other element is calculated. If the result agrees with the known value of that element, the assumed value of δ was correct. If not, a new trial value of δ is chosen and another iteration is accomplished.

Note that, since trigonometric functions are determined by the geometric size of an angle and not by the unit of measure:

$$\sin \delta/2 = \sin \Delta/2$$

$$\cos \delta/2 = \cos \Delta/2$$

and

$$\tan \delta/2 = \tan \Delta/2$$

If the calculator used can be operated in radian mode, the equations can be used as written. If not (as with the HP35), δ must be converted into degrees before taking trigonometric functions.

It is true that The Giddings Method is a brute force method. The power and speed of the HP series of calculators, however, make this objection trivial. The mathematics involved in deriving equations (3) through (13) from the basic curve equations (equations (1.1) through (1.5)) is of high school variety, so equations (3) through (13) need not be memorized or looked up. Finally, anyone familiar with the HP55 or 65 can easily write the programs for the solution of these equations, thereby making successive iterations almost effortless. Then too, in real life approximate values of Δ and/or R will be known from field observations or scale drawings, so a relatively accurate trial value will be available as a starting point.

For those who are interested, my HP55 programs for the iterative solution of equations (7), (8), (10), (11), (12), and (13) are included.

By the way, we require all of our students in Surveying Technology (first year college) to solve all 21 cases with the aid of a hand-held electronic calculator.

1. Switch to *ON*
2. Switch to *RUN*
3. *BST*
4. Switch to *PRGM*
5. Write one of the following programs as required:

Pointer Index	Given C^* and E Eq (7)	Given T^* and M Eq (8)	Given L^* and T Eq (10)	Given L^* and E Eq (11)	Given L^* and M Eq (12)	Given L^* and C Eq (13)
01	*STO*	*STO*	*STO*	*STO*	*STO*	*STO*
02	1	1	1	1	1	1
03	*RCL*	*RCL*	*RCL*	*RCL*	*RCL*	*RCL*
04	2	2	2	2	2	2
05	2	*RCL*	*RCL*	*RCL*	*RCL*	*RCL*
06	÷	1	1	1	1	1
07	*RCL*	2	÷	÷	÷	÷
08	1	÷	*STO*	*STO*	*STO*	*STO*
09	2	f	3	3	3	3
10	÷	tan	*RCL*	*RCL*	1	*RCL*
11	f	÷	1	1	*RCL*	1
12	sin	*STO*	2	2	1	2
13	÷	3	÷	÷	2	÷
14	*STO*	1	f	f	÷	f
15	3	*RCL*	tan	cos	f	sin
16	*RCL*	1	x	1/x	cos	x
17	1	2	*GTO* 00	1	–	2
18	2	÷		–	x	x
19	÷	f		x	*GTO* 00	*GTO* 00
20	f	cos		*GTO* 00		
21	cos	–				
22	1/x	x				
23	1	*GTO* 00				
24	–					
25	x					
26	*GTO* 00					

*See step 10.

6. Switch to *RUN*
7. *BST*

   If the arc length, L, is one of the known curve elements, execute steps 8 and 9 to go into radian mode. If L is not one of the known curve elements, omit steps 8 and 9 and pass directly to step 10.
8. f
9. *RAD*
10. Key in the value of the given element marked *.
11. *STO*
12. 2
13. Key in the trial value of Δ (in degrees) or the trial value of δ (in radians).
14. *R/S*
15. Go back to step 13 and try a revised value of Δ or δ. After Δ or δ has been determined to the required accuracy, go on to step 16.
16. *RCL*
17. 3

   The number read in the display is the value of R.

PETER A. STEEVES, Grad. Student Surveying Engineering, University of N.B.—After reading articles on the "Th Function" and other methods to solve the circular curve problem given the arc length and long chord, I decided to study the problem and came up with the enclosed solution.

Curve Solution Given Arc and Long Chord

Let $\theta = \Delta/2 \quad c = \text{chord} \quad A = \text{arc}$

then,

$$\theta = [10\times[1 - [1.2\,C/A - 0.2]^{1/2}]]^{1/2} \text{ radians} \quad (1)$$

$$d\theta \; \frac{c\theta - A\sin\theta}{c - A\cos\theta} \text{ radians} \quad (2)$$

$$\Delta/2 = \frac{A}{c}\,[\sin\theta\cos d\theta - d\theta\cos\theta] \text{ radians} \quad (3)$$

The $\Delta/2$ obtained from equations 1, 2 and 3 is accurate to 1 arc-sec. where $0° \leqslant \Delta/2 \leqslant 90°$. The equations were programmed on the HP65 where it took 7 seconds to compute $\Delta/2$.

Derivation

$$R = \frac{c}{2\sin\theta} = \frac{A}{2\theta}$$

$$\frac{\sin\theta}{\theta} = \frac{c}{A}$$

Expressing $\sin\theta/\theta$ by a Maclaurin Series,

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \cdots .$$

then

$$1 - \frac{\theta^6}{6} + \frac{\theta^4}{120} \approx \frac{c}{A}$$

$$\theta^4 - 20\,\theta^2 + 120(1 - c/A) = 0$$

$$\theta^2 = \frac{20 \pm (400 - 480(1 - c/A))^{1/2}}{2}$$

$$= 10(1 \pm (1.2\,c/A - 0.2)^{1/2})$$

since

$$\theta \leqslant \frac{\pi}{2}$$

$$\theta^2 \leqslant 2.47$$

$$\therefore \theta^2 = 10\times(1 - (1.2\,c/A - 0.2)^{1/2})$$

$$\theta = (10\times(1 - (1.2\,c/A - 0.2)^{1/2}))^{1/2}$$

This equation will always give an approximation for θ which is too large (0.5% at 90°)

Now we may write,

$$\frac{\sin(\theta - d\theta)}{\theta - d\theta} = \frac{\sin\theta\cos d\theta - \sin d\theta\cos\theta}{\theta - d\theta} = \frac{c}{A}$$

since $d\theta \longrightarrow 0\,(00° - 25')$ at 90°

we may write,

$$\sin d\theta \approx d\theta$$

$$\cos d\theta \approx 1$$

$$\frac{\sin\theta - d\theta\cos\theta}{\theta - d\theta} = \frac{c}{A}$$

$$d\theta = \frac{c\theta - A\sin\theta}{c - A\cos\theta}$$

Now we have a better approximation for $\cos d\theta$

$$\frac{\Delta}{2} = \frac{A}{c}(\sin\theta\cos d\theta - d\theta\cos\theta)$$

Corrigendum for— 'Length Ratios and Scale Unknowns on Trilateration' by T. Vincenty

Published in 'Surveying and Mapping,' Vol. XXXV, No. 3, Sept. 1975, pp. 245–250

On page 245, left column, bottom, a minus symbol was omitted from the second equation of the sequence. The equation should read:

$$a_1 = -r\Delta x_1/s_1^2$$

Therefore, the section in which the corrigendum is made should read:

$$a_o = r(\Delta x_1/s_1^2 - \Delta x_2/s_2^2) \qquad b_o = r(\Delta y_1/s_1^2 - \Delta y_2/s_2^2)$$

$$a_1 = -r\Delta x_1/s_1^2 \qquad b_1 = -r\Delta y_1/s_1^2$$

$$a_2 = r\Delta x_2/s_2^2 \qquad b_2 = r\Delta y_2/s_2^2.$$