Determination of Position with Intersecting Circles

By DAVID F. MATTSON
Senior Highway Technician
Minnesota Highway Department

FOREWORD—The intersecting circle technique for determination of position can result in great savings in time and equipment. Usually only one assistant is needed and conceivably the process could be performed single-handedly, especially if natural sights of known position are available. In many situations, no linear measurements are needed, thus eliminating the need for expensive electronic measuring instruments.

The accuracy of position, determined with intersecting circles, is comparable to conventional traversing results, even when no linear measurements are made. By roughly plotting the coordinate positions of the involved points and visually inspecting the intersection angles between circles, an analysis of "strength of figure can be easily made graphically." A "flat" intersection angle would obviously be given less weight than a "strong" one.

The beauty of the intersecting circle method is that one man with a reliable transit or theodolite can get as good results, in many situations, as a fully equipped survey crew using expensive electronic instruments.

Intersecting Circles

Horizontal position can be conveniently and accurately determined with interesting circles by using a modern electronic calculator and a reliable coordinate system. One of the simplest programs for a desk-type electronic calculator is the solution of two intersecting circles, given the coordinates of the radius points and the length of radii. The calculator will compute coordinates for the two intersection points, and the operator can decide which to use by visually inspecting the layout.

Circles can be defined by either angular or linear measurements. Position can be determined by using "angular circles" only, "distance" circles only, or a combination of both. Angular circles are advantageous if electronic measuring equipment is not available.

Circles Defined by Angle

An angular circle is defined by measuring the angle between two points of known coordinate positions from a third point of unknown position.

Figure 1 shows some of the sections of a simple curve. The angle between the beginning and end of the curve is the same for any point on the curve. This angle (*APB*), when subtracted from 180°, equals half the central angle (*ARB*). The central angle is the angle subtended at the radius point between the two points of known coordinate positions, *A* and *B*. *P* is the point of occupation and its position is unknown. The length and azimuth of chord *AB* can be easily calculated by taking the difference in coordinates between *A* and *B*. The radius point coordinates can be calculated by solving the right triangle formed by half the chord length and half the central angle. The unknown point does not have to be located between the known points.

In Figure 2 the angular circle is defined by the distance and azimuth between *A* and *B* and the angle measured at *P* (*APB*). Angle *APB* is equal to half the central angle. In this case the angle between the known points is not subtracted from 180°. Thus, a circular curve can be defined by considering the difference in coordinates between two known points and by measuring the angle between them in the field. If two circular curves are defined by this method, one of the

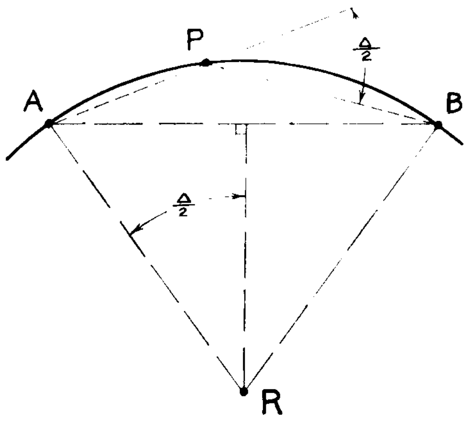

FIGURE 1.

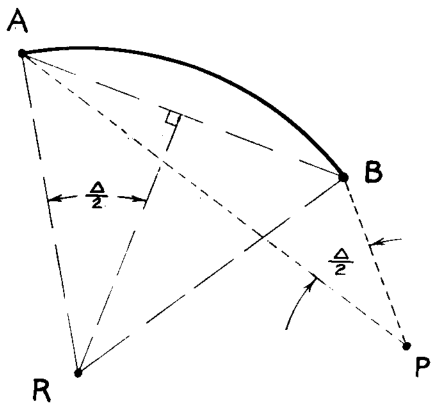

FIGURE 2.

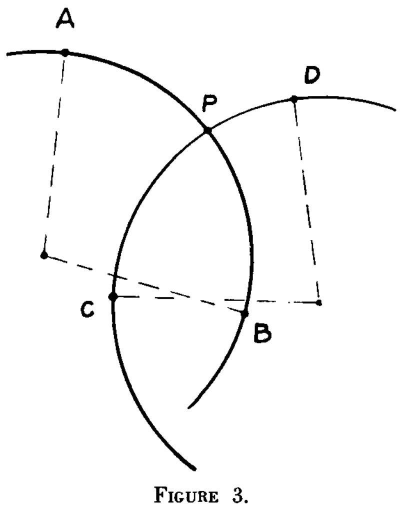

FIGURE 3.

points of intersection is the position of the unknown occupation point. Figure 3 shows an ideal situation in which two circular curves are defined by angles only. Note that in this situation the angle of intersection between the two curves is about 90°, which is the strongest intersection angle possible.

In addition to the circular curves in Figure 3, there are several other possible combinations of circles as illustrated in Figure 4. The additional circles are defined by measuring the angles between all four known points (*APC, APD, BPC,* and *BPD*). Using all the circles in various combinations results in 15 solutions to the position of unknown point *P*.

The position of an unknown point can be determined by averaging the intersection results of various circles with strong intersection angles. A layout of the circles (drawn approximately to scale) will show graphically which circle combinations are best. Intersecting circles with small intersection angles should not be considered. In Figure 4, for example, the intersection of the two circles defined by *APD* and *BPC* should not be considered. Field test results in Appendix A support this statement.

Horizontal position can be determined accurately with only three known points if there is a strong strength of figure as in Figure 5. The strength of figure can be easily analyzed graphically by noting the intersection angles between circles. If a poor strength of figure exists, a linear measurement to one or more of the known points is essential.

Circles Defined by Distance

Since a circle is defined by radius point coordinates and length of radius, a distance circle is defined when a linear measurement is made to a known point. The linear measurement is the radius length and the known point is the radius point. An advantage to using distance circles is that only two known points are needed. A disadvantage is that electronic measuring equipment is needed. Figure 6 is an example of position determination with intersecting distance circles. In this case, the intersection angle between the two circles is reasonably strong. An accurate position will result if the linear measurements are correct. A good check on the position can be made with an angular circle defined by angle *APB*. In this case, the angular circle passing through *A*, *P*, and *B* makes a strong intersection with either distance circle.

Angular Circles and Distance Circles Combined

A good example of the need for using angular circles and distance circles together is shown in Figure 7. In this case, each method is dependent on the other for successful results. The intersection angle between the two distance circles defined by radius points *A* and *B* is so flat that the computed position of point *P* would probably have a large error. The angular circle method alone will not work either, since there are only two known points available. An accurate determination of position can be made by intersecting each distance circle with the angular circle defined by angle *APB*. The angle of intersection between the angular circle and either of the distance circles is very strong in this example.

The solution of intersecting circles becomes complicated when several observations are measured with both angles and distances. Figure 8, for example, is identical to Figure 5 except that distances, as well as angles, were measured to the known points. There are now six circles to be considered, resulting in 15 different combinations of intersecting circles, compared to only three combinations in Figure 5. The best combinations of circles can be determined by graphically analyzing various intersection angles and radius lengths. Flat intersection angles should not be given much weight. Generally, three circles are sufficient if the coordinate system is reliable and the known points have a small standard deviation from true position. Excellent field test results were achieved on a highway project near St. Paul, Minnesota using three circles. The standard deviation from true position of the known points was 0.10 foot. The field test results can be found in Appendix A.

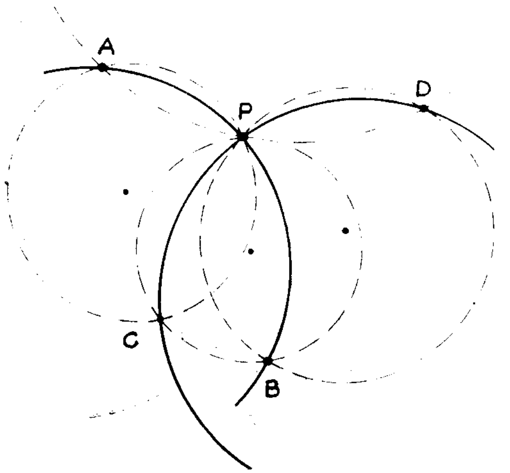

FIGURE 4.

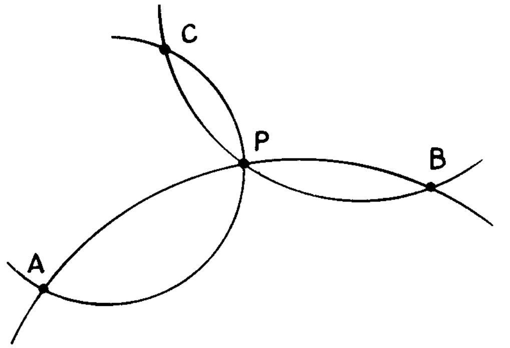

FIGURE 5.

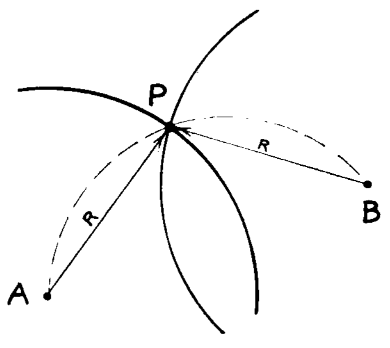

FIGURE 6.

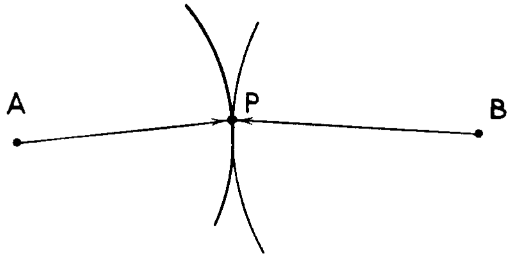

FIGURE 7.

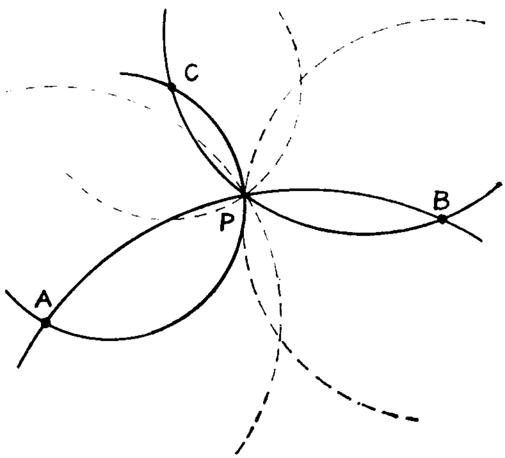

FIGURE 8.

Field Tests

Field tests were made on the above-mentioned project in situations where it became necessary to set new control points and to determine their coordinate positions from nearby known points. The intersecting circle method was used as a supplement to the conventional method. With the conventional method, an unknown point is traversed through from two points of known position. The closing error is adjusted by the compass rule. Two additional known points are needed for beginning and closing azimuths when using the conventional method. The coordinates determined with conventional methods were used as a measure of the intersecting circle results. Little or no extra field work was needed to determine the intersecting circles. The office work was greatly expedited by using a desk-type electronic calculator which was programmed to solve the parts of angular circles. Another program solved the intersection points of two intersecting circles. The two programs were separated to aid in the analysis of various circle combinations when several circles are available. The programs can be found in Appendix B.

While field tests can be accomplished easily requiring only one assistant, the intersecting circle system can be analyzed on the drawing board by creating a hypothetical coordinate system with a built-in standard deviation. Various circle combinations can be tested with allowances made for the standard deviation of the "known" points. Instrument errors also should be considered. Field tests are preferable, however, if a project such as the one mentioned above is available.

Economical Value of Intersecting Circles

Results of the field tests in Appendix A indicate that the intersecting circle method could have been used instead of conventional methods on that particular project. The use of intersecting circles can result in great savings in time and equipment. Field test 2 in Appendix A required three instrument set-ups and six target set-ups to determine the position of the unknown point when conventional methods were used. The intersecting circle technique required only one instrument set-up and two target set-ups. The results were nearly identical.

APPENDIX A

Field Test Results

Field Test 1

In Figure A the position of point *P* was determined with conventional methods, i.e., traversing through the unknown point from known points. There were instrument set-ups at *A*, *P*, and *B*, and target set-ups at *A*, *P*, *B*, *m*, and *n*. Sights were needed at points *m* and *n* for beginning and closing azimuths. The closing error was adjusted by the compass rule. A Wild T–2 theodolite and a Hewlett Packard 3800 distance-measuring instrument were used for the measurements.

With the intersecting circle method, one instrument set-up (at *P*) and three target set-ups (at *A*, *B*, and *C*) were used. Only two target set-ups were needed, but an extra target was set up at point *C* to test intersecting circles defined by angles only. The angles and distances shown are unadjusted.

The coordinates of point *P* are the results of conventional methods and were used as a comparison with the results of various combinations of in-

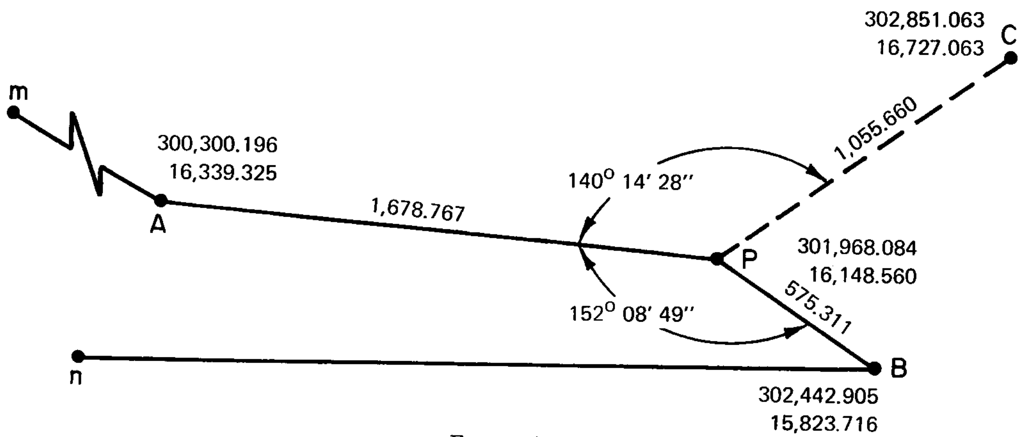

FIGURE A.

tersecting circles. Table 1 shows the results of the fifteen possible circle intersections in Figure A. Capital letters represent angular circles and small letters represent distance circles. *APB*, for example, is the angular circle passing through *A*, *P*, and *B*, and "a" is the distance circle defined by the distance from *P* to *A*. Point *C* appears to be out of position.

Table 1

Circle Combinations	*x*	*y*	Error
APB and APC	301,968.124	16,148.543	0.043 ft.
APB and BPC	301,968.124	16,148.543	0.043 ft.
APC and BPC	301,968.124	16,148.543	0.043 ft.
a and b	301,968.091	16,148.577	0.018 ft.
a and c	301,968.080	16,148.480	0.080 ft.
b and c	301,968.053	16,148.521	0.050 ft.
APB and a	301,968.089	16,148.561	0.005 ft.
APB and b	301,968.083	16,148.564	0.004 ft.
APB and c	301,967.994	16,148.610	0.103 ft.
APC and a	301,968.086	16,148.530	0.030 ft.
APC and b	301,968.051	16,148.519	0.053 ft.
APC and c	301,968.054	16,148.519	0.051 ft.
BPC and a	301,968.095	16,148.611	0.052 ft.
BPC and b	301,968.103	16,148.594	0.039 ft.
BPC and c (bad ints.)	301,968.287	16,148.164	0.445 ft.
Traverse Results	301,968.084	16,148.560	0.000 ft.

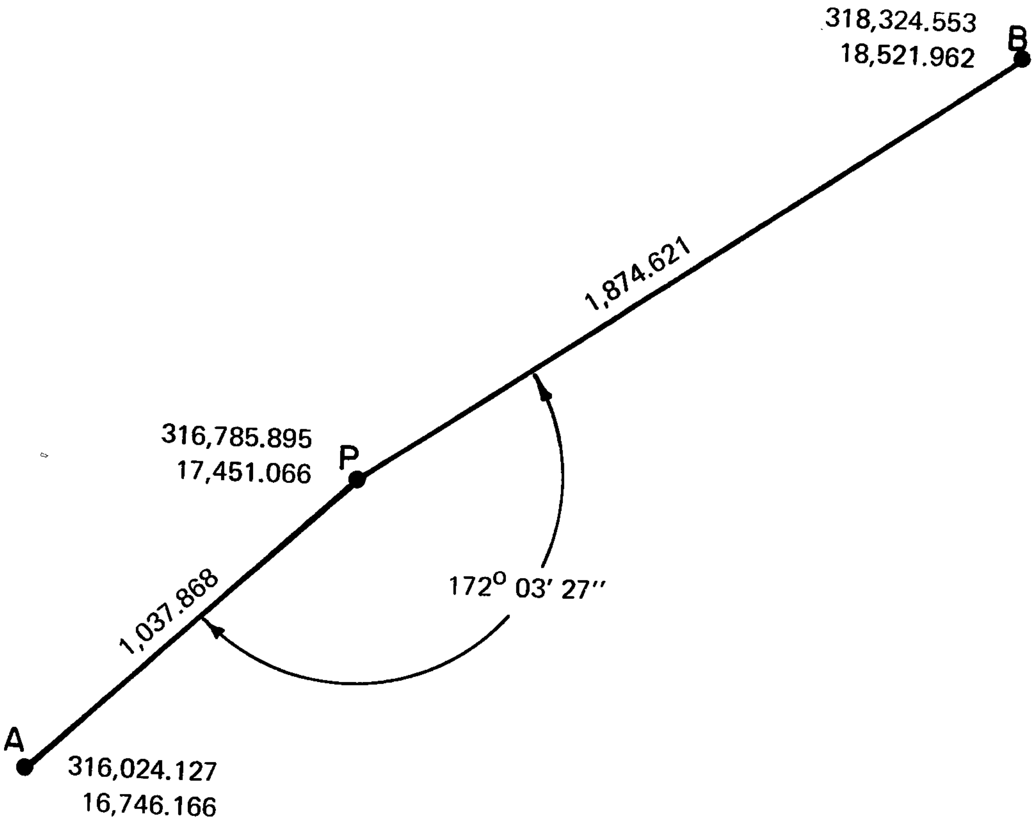

FIGURE B.

Field Test 2

The results of field test 1 indicate that an extra observation is unnecessary when both types of measurements are used together. The results of field test 2 support that premise. In field test 2, no extra field work was necessary to test the intersecting circle method. As in the first field test, the coordinates of the unknown point are the results of traversing through it and were used as a measure of the intersecting circle results. Figure B shows the layout of the points and Table 2 gives the results of the various intersecting circles. The angle and the two distances were measured from point *P* and are unadjusted.

The two intersecting distance circles did not produce very good results because of the weak intersection angle between them; therefore they should not be given much weight.

Table 2

Circle Combinations	x	y	Error
APB and a	316,785.889	17,451.070	0.007 ft.
APB and b	316,785.908	17,451.086	0.024 ft.
a and b	316,786.013	17,450.936	0.175 ft.
Traverse Results	316,785.895	17,451.066	0.000 ft.

Field Test 3

In this test the unknown point was not traversed through. The position was determined with intersecting circles exclusively, because of the

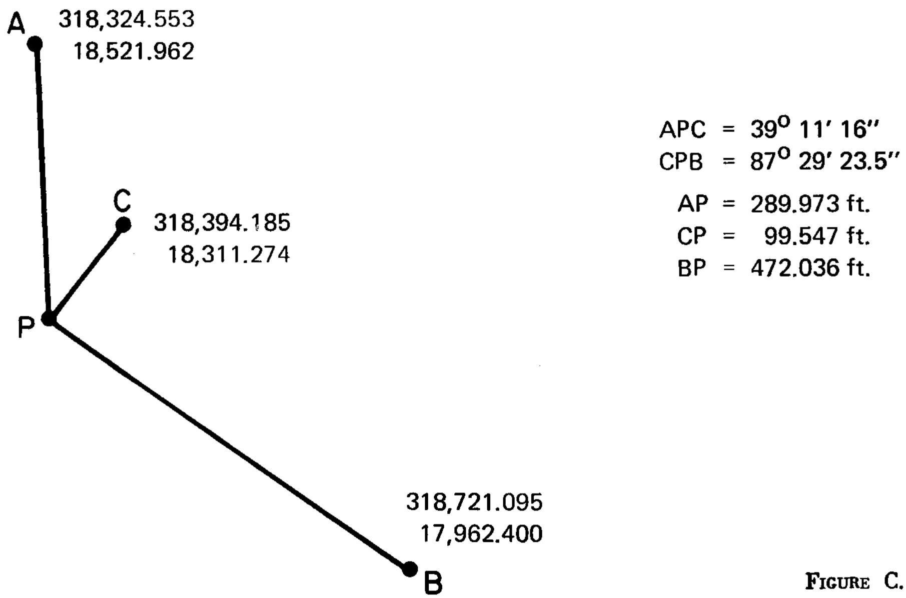

FIGURE C.

Table 3

Circle Combinations	x	y	Error*
APB and APC	318,333.771	18,232.158	0.016 ft.
APB and BPC	318,333.771	18,232.158	0.016 ft.
APC and BPC	318,333.770	18,232.158	0.015 ft.
a and b	318,333.796	18,232.136	0.040 ft.
a and c	318,333.717	18,232.134	0.044 ft.
b and c	318,333.746	18,232.175	0.029 ft.
APB and a	318,333.780	18,232.136	0.026 ft.
APB and b	318,333.772	18,232.155	0.015 ft.
APB and c	318,333.757	18,232.192	0.043 ft.
APC and a	318,333.738	18,232.135	0.024 ft.
APC and b	318,333.696	18,232.104	0.077 ft.
APC and c	318,333.769	18,232.157	0.014 ft.
BPC and a	318,333.759	18,232.135	0.014 ft.
BPC and b (NG)	318,333.848	18,232.321	0.194 ft.
BPC and c	318,333.770	18,232.157	0.014 ft.
MEAN	318,333.758	18,232.149	0.000 ft.

* Error from the mean of 14 intersections. The one marked *NG* should not be considered because of the weak angle of intersection.

successful results of the other tests. (*See* Figure C.) The unknown position was determined with one instrument set-up and three target set-ups. The field work required about an hour. The 15 intersecting circle combinations can be analyzed by examining Table 3.

Field Test 4

In Figure D, the position of point *P* was determined with conventional traversing methods. The distance from *A* to *P* was not measured. One extra angle was measured (*APB*) to test intersecting circles. The measurements in Figure D are unadjusted field measurements made from point *P*.

The angular circles, by themselves, produced poor results in this test because of the small intersection angles between them. Table 4 shows the results.

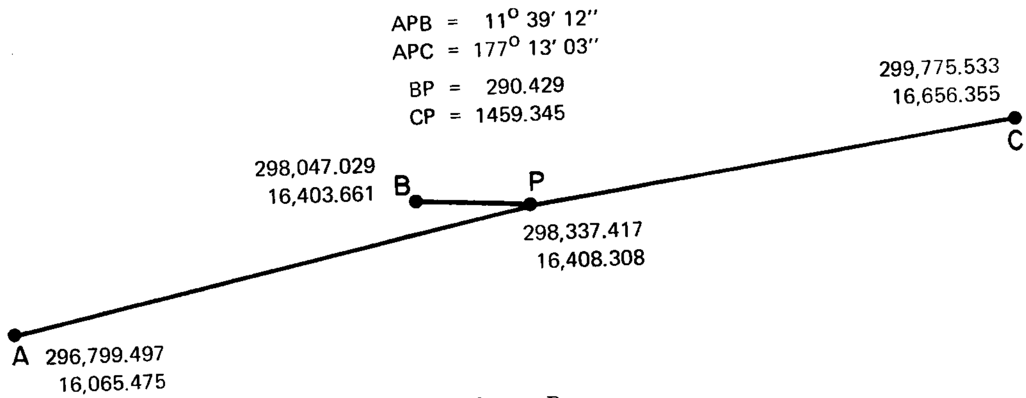

FIGURE D.

Table 4

Circle Combinations	x	y	Error
APC and ABP	298,337.228	16,408.293	0.190 ft.
APC and BPC	298,337.231	16,408.294	0.187 ft.
ABP and BPC	298,337.220	16,408.294	0.198 ft.
APC and b	298,337.420	16,408.331	0.023 ft.
APC and c	298,337.419	16,408.331	0.023 ft.
ABP and b	298,337.421	16,408.288	0.020 ft.
ABP and c	298,337.426	16,408.288	0.022 ft.
BPC and b	298,337.421	16,408.302	0.007 ft.
BPC and c	298,337.424	16,408.302	0.009 ft.
b and c	298,337.421	16,408.322	0.015 ft.
Traverse Results	298,337.417	16,408.308	0.000 ft.

APPENDIX B

Computer Programs for Hewlett-Packard 9100B

Program 1

Program 1 solves the two points of intersection of two intersecting circles when the radius point coordinates and length of radii are entered. Program 1 shows steps in solution.

The correct intersection point is decided arbitrarily by the calculator operator.

Procedure

(1) Set switches to DEGREES, FIXED POINT, POWER ON, and RUN.
(2) Enter side A of program card at +00.
(3) Set decimal wheel to 3.

Press	*Display*	*Register*	*Enter*	*Comments*
CONT				
	1.	Z	X1	Enter radius point coordinates and radius length of first circle.
	1.	Y	Y1	
	1.1	X	R1	
CONT				
	2.	Z	X2	Enter radius point coordinates and radius length of second circle.
	2.	Y	Y2	
	2.2	X	R2	
CONT				
	3.	Z		*Coordinates of one of the intersection points*
	X	Y		
	Y	X		
CONT				
	4.	Z		*Coordinates of the other intersection point*
	X	Y		
	Y	X		

See Program 1 notes on following page.

Program 2

When an angular circle is used, it is necessary to calculate the radius point coordinates and length of radius. These results can then be entered in the first program.

When using program 2, the field angle between the two known points must be subtracted from 180°. If the field angle is greater than 180°, it must be reduced by 180°. This rule must be followed even if the unknown point is not between the known points. Some examples are needed in order to clarify this rule. The correct angle to enter in the program is represented by ⊖.

Program 2 notes show steps in procedure.

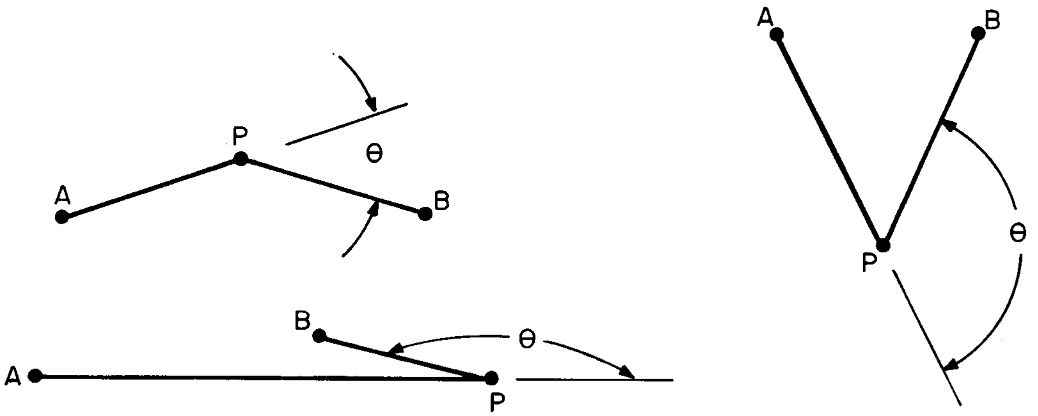

FIGURE E.

Program 1.—Notes

Title

Step	Key	Display x	Display y	Display z
00	CLR	0	0	0
1	1			
2	UP			
3	UP			
4	1			
5	.			
6	1	1.1	1	1
7	STOP	R1	Y1	X1
8	X to			
9	9	R1		
a	Y to			
b	b		Y1	
c	DN ↓			
0d	Y to			
10	a		X1	
1	X ⇄ Y	X1	Y1	
2	ACC−			
3	2			
4	UP			
5	UP			
6	2			
7	.			
8	2	2.2	2	2
9	STOP	R2	Y2	X2
a	ROLL ↓			
b	X ⇄ Y			
c	ACC+			
1d	RCL	X2−X1	Y2−Y1	
20	POLAR	Dist 2-1	±θ 2-1	R2
1	X to			
2	c	Dist 2-1		
3	Y to			
4	d		±θ 2-1	
5	DN ↓		R2	
6	Y to			
7	8		R2	
8	CLR	0	0	0
9	c	Dist 2-1		
a	UP			
b	X from			
c	9	R1		
2d	ACC+	R1	Dist 2-1	

Step	Key	Display x	Display y	Display z
30	POLAR			
1	UP			
2	X		$R1^2+R_1^2$	
3	X from			
4	8	R2		
5	UP			
6	X		$R2^2$	
7	DN ↓			
8	X ⇄ Y			
9	−			
a	2			
b	÷			
c	e	D_2-1		
3d	÷			
40	f	R1		
1	÷			
2	DN ↓			
3	CHS			
4	ARC			
5	Cos X	"C"		
6	UP			
7	CLR X	0		
8	UP			
9	Roll up	"C"	0	0
a	X to			
b	9	"C"		
c	f	R1		
4d	X to			
50	c			
1	CLR	0	0	0
2	YE			
3	9		"C"	
4	Y to			
5	9			
6	d	±θ		
7	+		Ang	
8	c	R1		
9	Rect.	ΔX	ΔY	
a	Acc+			
b	b	Y1		
c	UP			
5d	9	X1		

Step	Key	Display x	Display y	Display z
60	ACC+	X int	Y int	
1	RCL			
2	UP			
3	3			
4	Roll ↓			
5	X ⇄ Y	Y int	X int	3
6	IFF			
7	7			
8	6			
9	STOP	Y int	X int	3
a	CLR	0	0	0
b	SET			
c	YE			
6d	9		"C"	
70	DN ↓	"C"		
1	CHS	−"C"		
2	UP		−"C"	
3	Go to			
4	5			
5	6			
6	UP			
7	4			
8	Roll ↓			
9	STOP	Y int	X int	4
a	Go To			
b	0			
c	0			
7d	END			

	+ Storage	−
f	Acc ±	
e	Acc ±	
d	± θ 2-1	
c	R1	
b	Y1	
a	X1	
9	"C"	
8	R2	
7	Program	
6	"	
5	"	
4	"	
3	"	
2	"	
1	"	
0	"	

Procedure

(1) Set switches to DEGREES, FIXED POINT, POWER ON, and RUN.
(2) Enter side A of program card at +00 and side B at -00.
(3) Set decimal wheel to 3.

Press	*Display*	*Register*	*Enter*	*Comments*
CONT				
	.1	Z	Deg.	
	.1	Y	Min.	Enter ⊖
	.1	X	Sec.	
CONT				
	0	Z		Enter coordinates of easternmost known point
	2.	Y	X2	
	2.	X	Y2	
CONT				
	0	Z		
	1.	Y	X1	Enter coordinates of the other known point.
	1.	X	Y1	
CONT				
	0	Z		
	0	Y		
	1.2	X	1 or 2	Enter the location of the unknown point with respect to the extended line between the two known points: 1 if above or 2 if below.*
CONT				
	R	Z		Radius length.
	X	Y		Radius point coordinates.
	Y	X		
CONT				
	2.1	Z		
	2.1	Y		Distance between known points.
	distance	X		
CONT				
	Deg.	Z		
	Min.	Y		Azimuth between known points.
	Sec.	X		

* An explanation is in order for identifying the location of the unknown point with respect to the extended line passing through the two known points. This line must be thought of as being extended infinitely in both directions. Figure F should clarify this rule.

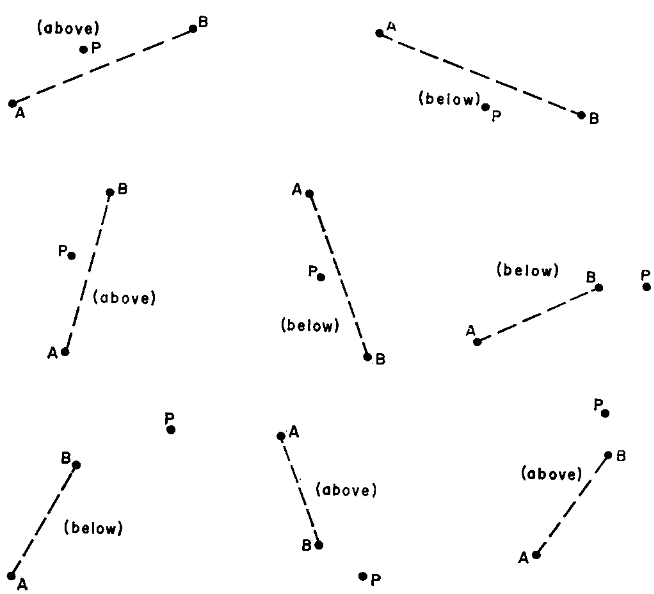

FIGURE F.

Program 2—Notes

Title ____________________

Step	Key	Display x	Display y	Display z
00	·			
1	1			
2	↑			
3	↑	.1	.1	.1
4	Stop	sec	min	deg
5	Roll ↑			
6	X to			
7	d			
8	6			
9	0			
a	÷			
b	Roll ↓			
c	+			
d	↓			
10	X⇄Y			
1	÷			
2	d			
3	+		θ	
4	Y to			
5	8		θ	
6	9			
7	0	90		
8	X⇄Y		90	
9	−	θ	90-θ	
a	Y to			
b	9		90-θ	
c	clear	0	0	0
d	2			
20	↑	2.	2.	0
1	stop	Y2	X2	
2	Y to			
3	a			
4	X to			
5	b			
6	Acc−			
7	0			
8	↑			
9	1			
a	↑	1.	1.	0
b	stop	Y1	X1	
c	Acc+			
d	RCL			

Step	Key	Display x	Display y	Display z
30	to Polar	L.C.	−Az	
1	X to			
2	d	L.C.		
3	3			
4	6			
5	0	360	−Az	
6	+		Az	
7	Y to			
8	c			
9	clear	0	0	0
a	1			
b	·			
c	2	1.2	0	0
d	stop	1 or 2		
40	Roll ↑			
1	1	1	1 or 2	
2	X=Y			
3	9			
4	3			
5	c	Az		
6	↑		Az	
7	X from			
8	9	90-θ		
9	+		Az	
a	3			
b	6			
c	0	360		
d	↑	360	360	Az
50	Roll ↓	360	Az	360
1	X<Y			
2	Arc			
3	−			
4	0			
5	X>Y			
6	Roll ↓			
7	+			
8	Y to			
9	7		Az	
a	X from			
b	8	θ		
c	Sin X	sin θ		
d	↑			

Step	Key	Display x	Display y	Display z
60	2	2		
1	×	2	2sinθ	
2	d	L.C.		
3	X⇄Y	2sinθ	L.C.	
4	÷		R	
5	Y to			
6	6		R	
7	a	X		
8	↑			
9	b	Y	X	
a	Acc+			
b	X from			
c	7		RAz	
d	↑			
70	X from			
1	6	R	RAz	
2	Rect			
3	Acc+			
4	RCL	YR	XR	
5	Roll ↑			
6	X from			
7	6	RAD	YR	XR
8	Roll ↓			
9	Stop			
a	2			
b	·			
c	1	2.1		
d	↑	2.1	2.1	

+ Storage X		
f		X
e		
d	L.C. (chord)	
c	Az of L.C. (bet 180° & 360°)	
b	Y co-ord. of east point	
a	X co-ord. of east point	
9	90° − θ	
8	θ	
7	Az R	
6	Radius	
5		
4		
3		
2		
1		
0		

Program 2—continued

Title ______

Step	Key	Display x	Display y	Display z
80	↑	2.1	2.1	2.1
1	d			
2	stop	L.C.	2.1	2.1
3	clear			
4	c			
5	↑			
6	Int. X			
7	x to			
8	a			
9	—			
a	6			
b	0			
c	x			
d	x⇄y			
90	↑			
1	int. X			
2	X to			
3	b			
4	—			
5	↓			
6	x⇄y			
7	X			
8	a			
9	↑			
a	b			
b	Roll ↑			
c	stop			
d	Go to			
a0	+			
1	0			
2	0			
3	c	AZ		
4	↑		AZ	
5	x from			
6	9	90-θ		
7	ch sign	-90-θ	AZ	
8	Go to			
9	—			
a	4			
b	9			
c	END			
d				

Step	Key	Display x	Display y	Display z
00	clear			
1	Go to			
2	—			
3	0			
4	0			

Step	Key	Display x	Display y	Display z

	+ Storage	−
f		
e		
d		
c		
b		
a		
9		
8		
7		
6		
5		
4		
3		
2		
1		
0		

■