COORDINATE SYSTEMS AND THE THREE POINT PROBLEM

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Plane coördinates are not commonly used for local survey work in Canada. For many types of surveys a local, plane coördinate system would be of great use in simplfying the plotting and computation of points and in keeping records of the positions and locations of posts. When one has to compute a three point problem, the advantages of plane coördinates are very clearly seen.

Where a plane coördinate system is used, a first-order control should be established and all the minor control points directly or indirectly tied in to the first-order control. In general the best way to establish first-order control is by a triangulation system, either as a distinctive local net or as an increased density of points in a larger net. One of the first-order points should be used as the origin of the plane coördinate system. In urban areas it is advisable to establish the positions of all prominent points such as church spires and chimneys.

For the computation of plane coördinates from geographic positions, Special Publication No. 71 of the U.S. Coast and Geodetic Survey, "Relation between Plane Rectangular Coördinates and Geographic Positions", by Walter F. Reynolds, may be used. An area up to 30 miles by 30 miles may be covered by a single coördinate system without any noticeable distortion.

The X-Y coördinate system, with the two coördinate axes at right angles to one another, is the most common. The position of any point in the plane, formed by the two coördinate axes, is expressed in X and Y coördinates (see fig. 1). The following notation will be used throughout this article.

Let the COORDINATES of a point P be X_P and Y_P. The numerical values for X (or the abscissa) are considered positive to the right or east of the Y-axis. The numerical values for Y (or the ordinate) are considered positive to the north or above the X-axis. The centre or origin 0 of the coördinate system is given the values, $X_0 = 0$ and $Y_0 = 0$. Sometimes the origin of the system is given two large positive values, so that in the area no negative coördinate values will occur. The origin or centre of the plane coördinate system is the reference point as mentioned in the definition for a bearing given below. The origin is also the important point to be used in the computations to obtain plane rectangular coördinates from geographic positions.

Directions

The notation for directions observed from a point P to the points A, B, and C are *(PA)*, *(PB) and (PC)*, the parentheses indicating non-oriented values.

Bearings

The notation for the bearings for these directions are *PA, PB* and *PC*.

In the plane coördinate system one generally refers to bearings and not to azimuths.

A bearing *PA* is the angle, measured clockwise from north, between the direction at the point P, parallel to the meridian at the reference point 0 (centre of the coördinate system), and the direction at the point P to the point A.

An azimuth *PA* is the angle, measured clockwise from north, between the meridian at the point P and the direction at the point P to the point A.

The difference in value between a bearing and an azimuth is caused by the convergence, which is a function of the sine of the middle latitude and the difference in longitude between the point P and the reference point 0.

From figure 2 we have

$$\alpha_1 = AB \text{ and } \alpha_2 = BA \qquad AB + 180° = BA,$$

$$\tan \alpha_1 = \tan AB = \frac{X_B - X_A}{Y_B - Y_A} \text{ (1)}, \quad \tan \alpha_2 = \tan BA = \frac{X_A - X_B}{Y_A - Y_B} \text{ (2)}$$

Negative values for $X_A - X_B$ and $Y_A - Y_B$ indicate that the value for *BA* is between 180° and 270°.

By following this method of notation one can immediately see in which quandrant the value for the bearing is. Only when using bearings is the relationship $AB = 180° + BA$ true. As soon as we deal with azimuths we have to allow for convergence.

The length of the line AB can be computed from the value *AB*.

$$AB = \frac{X_B - X_A}{\sin AB} = \frac{Y_B - Y_A}{\cos AB} \tag{3}$$

A variation of the commonly used plane, rectangular coördinate system is the plane, rectangular, *homogeneous* coördinate system. In this system the three coördinates *X*, *Y* and Z are given for each point. The relationship between the commonly used coördinates and these homogeneous coördinates is expressed as follows:

$$X_P = \frac{\mathit{X}_P}{Z_P}, \quad Y_P = \frac{\mathit{Y}_P}{Z_P} \tag{4}$$

For points at infinity the values for Z are zero. This is one of the advantages of this coördinate system. Usually the following simple relation between the coördinate values is used:

$$\mathit{X}_P = X_P ; \ Y_P = \mathit{Y}_P ; \ Z_P = 1.$$

BARICENTRIC coördinates offer another interesting system, which is presented in the following paragraphs in some detail. They give an interesting and useful solution to the three point problem.

The equation of a straight line in two dimensional analytical geometry is

$$L = a_1X + a_2Y + a_3 = 0 \tag{5}$$

where a_1, a_2 and a_3 are constants.

The perpendicular distance from a point P (coördinates X_P and Y_P) to the line L = 0 is:

$$l_P = \frac{a_1X + a_2Y + a_3}{\sqrt{(a_1^2 + a_2^2)}} \tag{6}$$

Let:

$$a_1X + a_2Y + a_3 = 0 = L$$

$$b_1X + b_2Y + b_3 = 0 = M \qquad (7)$$

$$c_1X + c_2Y + c_3 = 0 = N$$

represent the equations of three straight lines, where a, b and c are constants and where the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

does not equal zero.

The three lines then do not intersect in a point, but form a triangle (see fig. 3). Take a point P with plane coördinates X_P and Y_P. For the distances l_P, m_P and n_P, which are the perpendicular distances from P to the lines $L=0$, $M=0$ and $N=0$, the following expressions exist:

$$l_P = \frac{a_1X_P + a_2Y_P + a_3}{\sqrt{(a_1^2 + a_2^2)}}$$

similarly,

$$m_P = \frac{b_1X_P + b_2Y_P + b_3}{\sqrt{(b_1^2 + b_2^2)}} \qquad (8)$$

and

$$n_P = \frac{c_1X_P + c_2Y_P + c_3}{\sqrt{(c_1^2 + c_2^2)}}$$

For the expression $a_1X_P + a_2Y_P + a_3$ write L_P, substitute in the first equation of (8) and obtain the equation:

$$l_P = \frac{L_P}{\sqrt{(a_1^2 + a_2^2)}}$$

$$m_P = \frac{M_P}{\sqrt{(b_1^2 + b_2^2)}} \qquad (9)$$

$$n_P = \frac{N_P}{\sqrt{(c_1^2 + c_2^2)}}$$

which lead to the following expressions for a point P:

$$L_P : M_P : N_P =$$

$$l_P \sqrt{(a_1^2 + a_2^2)} : m_P \sqrt{(b_1^2 + b_2^2)} : n_P \sqrt{(c_1^2 + c_2^2)} \qquad (10)$$

Similarly for a point U which will be called the Unit point,

$$L_U : M_U : N_U =$$

$$l_U \sqrt{(a_1^2 + a_2^2)} : m_U \sqrt{(b_1^2 + b_2^2)} : n_U \sqrt{(c_1^2 + c_2^2)} \qquad (11)$$

Assume for the unit point that $L_U : M_U : N_U = 1 : 1 : 1$, and divide (10) by (11) to obtain:

$$L_P : M_P : N_P = \frac{l_P}{l_U} : \frac{m_P}{m_U} : \frac{n_P}{n_U} \qquad (12)$$

L_P, M_P and N_P are called the triangular coördinates for the point P.

Take for the unit point U the centre of gravity of the triangle, formed by the three lines L=0, M=0 and N=0 (this triangle is called the basic triangle). The values for L_P, M_P and N_P are called the *baricentric* coördinates.

In figure 4 the baricentric coördinates of the points A, B and C (being the intersections of the lines L=0, M=0 and N=0) are

$$L_A : M_A : N_A = 3 : 0 : 0 = 1 : 0 : 0$$

$$L_B : M_B : N_B = 0 : 3 : 0 = 0 : 1 : 0.$$

$$L_C : M_C : N_C = 0 : 0 : 3 = 0 : 0 : 1.$$

Using baricentric coördinates the following is true:

$$l_U : m_U : n_U = h_A : h_B : h_C \text{ and as}$$

$$2 \text{ Area}_{ABC} = a\,h_A = b\,h_B = c\,h_C,$$

thus, $$h_A : h_B : h_C = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$$

so, $$l_U : m_U : n_U = \frac{1}{a} : \frac{1}{b} : \frac{1}{c} \qquad (13)$$

Substitute (13) in (12) to obtain

$$L_P : M_P : N_P = \text{Area}_{BCP} : \text{Area}_{CAP} : \text{Area}_{ABP} = a l_P : b m_P : c m_P \qquad (14)$$

Note that a negative sign indicates that a triangle is situated in the opposite direction to the triangle formed by the unit point.

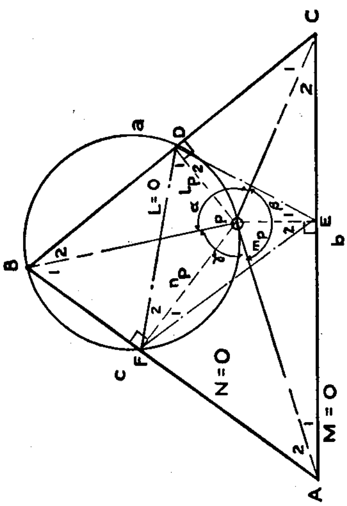

FIGURE 5

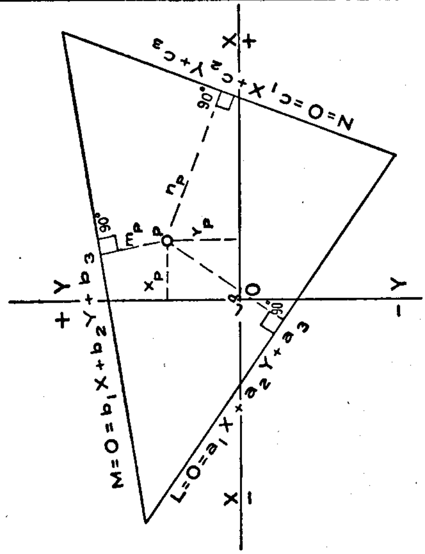

FIGURE 3

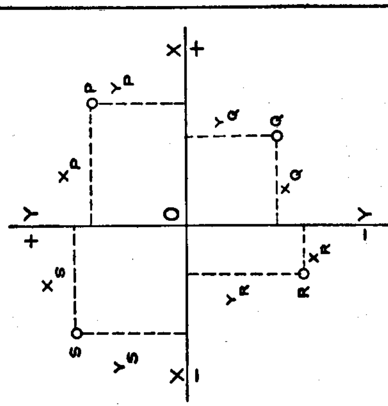

FIGURE 1

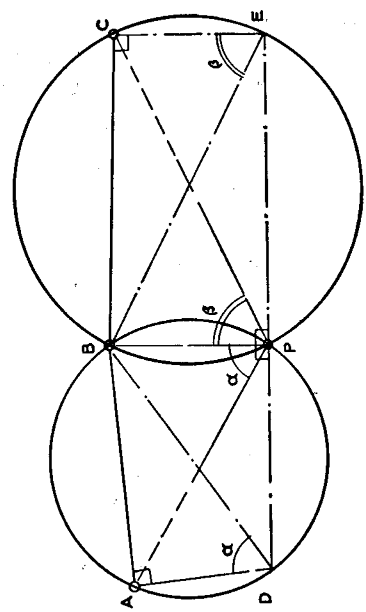

FIGURE 6

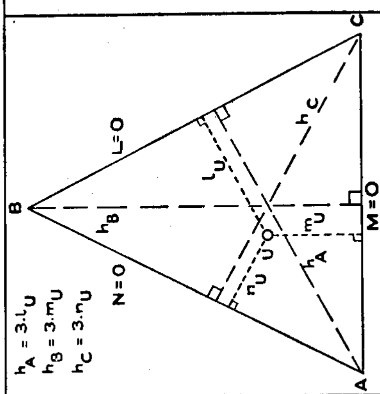

FIGURE 4

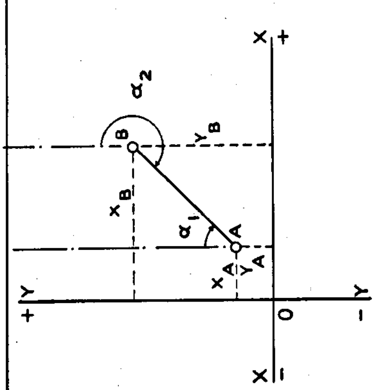

FIGURE 2

Using this basic information about the baricentric coördinates, one solution is derived for the three point problem. At a point P the directions (PA), (PB) and (PC) are observed to three known points A, B and C. The positions of these three points are given in plane rectangular coördinates. From P draw perpendiculars PD, PE and PF, respectively, to BC, CA and AB (see figure 5).

In the quadrilateral BDPF note that $\angle D = 90°$ and $\angle F = 90°$, so BDPF is a cyclic quadrilateral. Thus $\angle D_1 = \angle B_1$ (subtend same arc PF), and similarly in the qadrilateral DCEP, $\angle D_2 = \angle C_2$ (subtend same arc PE).

$\angle FDE = \angle D_1 + \angle D_2 = \angle B_1 + \angle C_2 = 180° - \gamma - A_2 + 180° - A_1 - \beta$, so $\angle FDE = 360° - (\gamma + \beta) - A = \alpha - A$.

Similarly, $\angle DEF = \beta - B$ and $\angle EFD = \gamma - C$.

The radius of the circle circumscribing the quadrilateral BFPD is BP/2 since $\angle D = 90°$ and is equal to the radius of the circumscribed circle of triangle BFD.

In the triangle BFD, FD = BP sin B, or $BP = \dfrac{FD}{\sin B}$; and similarly, $AP = \dfrac{EF}{\sin A}$ and $CP = \dfrac{DE}{\sin C}$.

Denote the radius of the circumscribed circle of the triangle DEF by r.

Then $DE = 2\,r \sin DFE = 2\,r \sin (\gamma - C)$.

Substituting this value of DE gives $CP = \dfrac{2\,r \sin (\gamma - C)}{\sin C}$

Similarly, $AP = \dfrac{2\,r \sin (\alpha - A)}{\sin A}$ and $BP = \dfrac{2\,r \sin (\beta - C)}{\sin B}$.

The area of the triangle ABP is ½ AP BP sin γ,

$$\text{or } 2\,\text{area}_{ABP} = AP\ BP\ CP\ \frac{\sin\gamma}{CP}$$

$$= \frac{AP\ BP\ CP}{2\,r}\ \frac{\sin C\ \sin\gamma}{\sin(\gamma - C)}$$

also

$$2\,\text{area}_{BCP} = \frac{AP\ BP\ CP}{2\,r}\ \frac{\sin A\ \sin\alpha}{\sin(\alpha - A)}$$

$$2\,\text{area}_{CAP} = \frac{AP\ BP\ CP}{2\,r}\ \frac{\sin B\ \sin\beta}{\sin(\beta - B)}$$

Substitute these values in (14) to obtain

$$L_P : M_P : N_P = \frac{\sin A\ \sin\alpha}{\sin(\alpha - A)} : \frac{\sin B\ \sin\beta}{\sin(\beta - B)} : \frac{\sin C\ \sin\gamma}{\sin(\gamma - C)}, \text{ or}$$

$$L_P : M_P : N_P = \frac{1}{\cot A - \cot\alpha} : \frac{1}{\cot B - \cot\beta} : \frac{1}{\cot C - \cot\gamma} \qquad (15)$$

In formula (15) the baricentric coördinates of P are expressed in terms of the angles A, B and C of the basic triangle ABC and the observed angles α, β and γ, where $\alpha = (PB) - (PA)$; $\beta = (PC) - (PB)$; $\gamma = (PA) - (PC)$.

The next step is the derivation of formulas for transferring *baricentric* coördinates to *homogeneous* rectangular coördinates.

Substitute the values $X = \frac{X}{Z}$ and $Y = \frac{Y}{Z}$ in equations (7)

$$\begin{aligned} L &= a_1 X + a_2 Y + a_3 Z \\ M &= b_1 X + b_2 Y + b_3 Z \\ N &= c_1 X + c_2 Y + c_3 Z \end{aligned} \qquad (16)$$

Expressing X, Y and Z in terms of L, M and N, we get the following equations:

$$\begin{aligned} X &= A_1 L + A_2 M + A_3 N \\ Y &= B_1 L + B_2 M + B_3 N \\ Z &= C_1 L + C_2 M + C_3 N \end{aligned} \qquad (17)$$

where A, B and C are constants (subject to the same conditions as a, b and c).

On comparing the coördinate value for three points in both coördinate systems, the values of the nine unknown coefficients can be obtained.

The three points to be used are A, B and C from the basic triangle.

	Homogeneous rectangular coörds.			Baricentric coörds.			
Point	X	Y	Z	L	M	N	
A	X_A	Y_A	1	1	0	0	
B	X_B	Y_B	1	0	1	0	(18)
C	X_C	Y_C	1	0	0	1	

Substitute these values in equations (17) to get

$$\begin{aligned} A_1 &= X_A & A_2 &= X_B & A_3 &= X_C \\ B_1 &= Y_A & B_2 &= Y_B & B_3 &= Y_C \\ C_1 &= 1 & C_2 &= 1 & C_3 &= 1 \end{aligned}$$

Substituting these results in equations (17) will give the following relations:

$$\begin{aligned} X &= L X_A + M X_B + N X_C \\ Y &= L Y_A + M Y_B + N Y_C \\ Z &= L + M + N \end{aligned}$$

Transferring from homogeneous to common rectangular coördinates (4), gives

$$X = \frac{L X_A + M X_B + N X_C}{L + M + N}$$

$$Y = \frac{L Y_A + M Y_B + N Y_C}{L + M + N} \qquad (20)$$

Substitute the values for L, M and N from equation (15) in (20) to obtain

$$X_P = \frac{\dfrac{X_A}{\cot A - \cot \alpha} + \dfrac{X_B}{\cot B - \cot \beta} + \dfrac{X_C}{\cot C - \cot \gamma}}{\dfrac{1}{\cot A - \cot \alpha} + \dfrac{1}{\cot B - \cot \beta} + \dfrac{1}{\cot C - \cot \gamma}} \qquad (21)$$

and a similar expression for Y_P.

To compute the values for the cotangents of the angles A, B and C the following derivation is used:

$$\cot A = \frac{\cos A}{\sin A} = \frac{b\ c \cos A}{b\ c \sin A} = \frac{b\ c \cos (AC - AB)}{2\ \text{area}_{ABC}}$$

$$= \frac{b \cos AC\ \ c \cos AB + b \sin AC\ \ c \sin AB}{2\ \text{area}_{ABC}}$$

$$= \frac{(Y_C - Y_A)(Y_B - Y_A) + (X_C - X_A)(X_B - X_A)}{2\ \text{area}_{ABC}}$$

Put $Y_A - Y_B = y_3$; $Y_B - Y_C - y_1$; $Y \ - Y_A = y_2$; and with similar notations for the X differences.

Then $$\cot A = \frac{y_2 y_3 + x_2 x_3}{-2\ \text{area}_{ABC}} \qquad (22)$$

with similar expressions for cot B and cot C.

From analytical geometry we have,

$$\begin{vmatrix} Y_A & Y_A & 1 \\ X_B & Y_B & 1 \\ X_C & Y_C & 1 \end{vmatrix} = \begin{vmatrix} X_A & Y_A & 1 \\ X_B - X_A & Y_B - Y_A & 0 \\ X_C - X_A & Y_C - Y_A & 0 \end{vmatrix} = \begin{vmatrix} -x_3 & -y_3 \\ x_2 & y_2 \end{vmatrix}$$

$$= x_2 y_3 - x_3 y_2 = -2\ \text{area}_{ABC}$$

Similarly, $-$ 2 $\text{area}_{ABC} = x_3y_1 - x_1y_3 = x_1y_2 - x_2y_1$.

As a check on the computations of the cotangent values the following expression is used:

$$\cot p \cot q + \cot q \cot r + \cot r \cot p = 1,$$
$$\text{where } p + q + r = n\,180^\circ \text{ for integral values of } n.$$

The three point problem is indeterminate when the three known points are concyclic. This method does not afford a solution if the three points are situated on a straight line, because cot A = cot B = cot C = infinity. The above method is specially adapted to the use of a desk calculator.

Finally, the CASSINI and COLLINS methods for solving the three point problem are given here. The Cassini method is based on the principle as used by L. F. Gregerson in the January, 1954, issue of the *Canadian Surveyor,* but the method of computation is different.

The Cassini Method

The coördinates of the three points A, B and C are given. The three directions (PA), (PB) and (PC) are observed at P. $(PB) - (PA) = \alpha$; $(PC) - (PB) = \beta$; (see fig. 6). The line DE is at right angles to the line PB. $\angle BDA = \angle BPA = \alpha$. The two angles subtend the same arc. Similarly $\angle CEB = \angle CPB = \beta$.

$$\begin{aligned} X_D - X_A &= AD \sin AD \\ &= AB \cot \alpha \sin (AB + 90^\circ) \\ &= AB \cot \alpha \cos AB \\ &= (Y_B - Y_A) \cot \alpha \end{aligned}$$

Put $Y_B - Y_A = y_1$, then $X_D = X_A + y_1 \cot \alpha$. (23)

Similarly when $Y_C - Y_B = y_2$ we have $X_E = X_C + y_2 \cot \beta$. (24)

The same method is used to compute Y_D and Y_E.

$$Y_D = Y_A + x_1 \cot \alpha \qquad (25)$$

$$Y_E = Y_C + x_2 \cot \beta \qquad (26)$$

$$x_1 = X_A - X_B \text{ and } x_2 = X_B - X_C$$

The coördinates of the points D and E are known.

Put $X_D - X_E = x$ and $Y_D - Y_E = y$ and $Y_D - Y_B = -(Y_B - Y_P) - (Y_P - Y_D)$ to obtain the following expression:

$$Y_D - Y_B = (X_B - X_P) \tan DE - (X_P - X_D) \cot DE \qquad (27)$$

Put $\tan DE = \frac{x}{y} = n$; $\cot DE = \frac{y}{x} = \frac{1}{n}$; and $n + \frac{1}{n} = N$.

Substitute these notations in (27) to get

$$Y_D - Y_B = (X_B - X_P)\,n - (X_P - X_D)\,\frac{1}{n}$$

$$X_P\left(n + \frac{1}{n}\right) = n\,X_B + Y_B + \frac{1}{n}\,X_D - Y_D$$

Hence
$$X_P = \frac{n\,X_B + \frac{1}{n}\,X_D + Y_B - Y_D}{N} \qquad (28)$$

$$Y_P = \frac{n\,Y_B + \frac{1}{n}\,Y_D + X_B - X_D}{N}$$

The use of X and Y coördinates affords a neat way of writing the different steps leading to the final solution. This method is also specially adapted to the use of a desk calculator.

The Collins Method

The COLLINS method is specially adopted to the use of logarithms. We have the same data as used for the Cassini method.

First the length and bearing of the line AB are computed. The angles A and D in triangle ABD are respectively equal to the angles DPB and BPA (angles subtending the same arc. See figure 7). The coordinates of the point D (called the auxiliary point by Collins) are computed as follows:

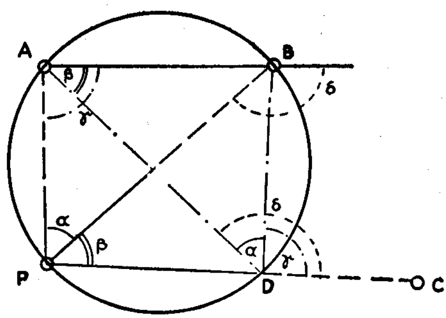

FIGURE 7.

In the triangle ABD, $\frac{AB}{\sin D} = \frac{AB}{\sin \alpha} = m$ (29)

and
$$BD = m \sin \beta \qquad \mathit{BD} = \mathit{AB} + \alpha + \beta$$
$$AD = m \sin (\alpha + \beta), \quad \mathit{AD} = \mathit{AB} + \beta \qquad (30)$$

leading to the following equations:

$$X_D = X_A + AD \sin \mathit{AD} \text{ or } X_D = X_B + BD \sin \mathit{BD}$$

and
$$Y_D = Y_A + AD \cos \mathit{AD} \text{ or } Y_D = Y_B + BD \cos \mathit{BD}$$

The bearing $PD = DC$ can be computed from $\tan PD = \tan DC = \frac{X_C - X_D}{Y_C - Y_D}$

Next the angles $\gamma = DC - DB$ and $\delta = DC - DA$

In the triangle ABP the angles A and B are respectively equal to γ and $180° - \delta$.

The coördinates of the point P can be obtained in a manner similar to the computation of X_D and Y_D.

$$AP = m \sin \delta$$

$$BP = m \sin \gamma$$

$$AP = AB + \gamma \text{ or } AP = 180° + PD - \alpha - \beta$$

$$BP = AB + \delta \text{ or } BP = 180° + PD - \beta$$

Thus

$$X_P = X_A + AP \sin AP \text{ or } X_P = X_B + BP \sin BP$$

and

$$Y_P = Y_A + AP \cos AP \text{ or } Y_P = Y_B + BP \cos BP$$

The principle in the last two solutions of the three point problem is to compute the bearing for one of the observed directions. This orients the observed directions. In both methods one or two circumscribed circles and auxiliary points are used to get this orientation.

The principle of the Collins method is used in the U.S. Coast and Geodetic Survey Special Publication No. 138 *"Manual of Triangulation Computation and Adjustment,"* pages 191 and 192, to derive the formulas used there.

"Attendant upon completion of the seaway will be economic benefits which engineers and economists have indicated will be staggering . . . The first and most important will be the removal of the 112 mile - 14′ bottleneck between Lake Ontario and Montreal which will permit extension of the peculiar characteristics of the Great Lakes navigation system . . . It is estimated that between 30 and 36 million tons of traffic will move through the seaway the first year of navigation, 1959. Of this some 10,000,000 tons will be in the form of iron ore from the Labrador fields . . . Another seaway benefit would be the savings in the cost of transporting grain, flour, coal, machinery and other commodities. It has been estimated that the saving in the cost of grain will be in the neighborhood of five cents a bushel . . . Up bound vessels with ore and other cargoes will find it of advantage to carry grain and other downbound cargoes making for a greater economy in the use of the vessels. It is estimated that this saving will amount to $30,000,000 a year . . . Careful survey of these and many other advantages have brought our economists to the conclusion that the St. Lawrence Seaway and Power development will benefit the Canadian economy to the extent of somewhere around $1,000,000,000 annually."

—Hon. Lionel Chevrier,
President of St. Lawrence Seaway Authority.