

Computing Missing Elements of a Polygon

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Introduction

Occasionally the surveyor encounters an old boundary or a closed traverse where two elements have been omitted. These elements may be two distances, one distance and one bearing, or two bearings. In several surveying textbooks the problems are discussed, and the method of solution is "brute force."

This paper will develop by means of algebraic and trigonometric relationships a set of formulas to simplify these computations.

General Relationship

[Formulae 1.01-1.14]

For any closed figure of n sides (the sides are numbered sequentially from 1 to n) the sum of the latitudes, Δy , and the sum of the departures, Δx , must equal zero. Or, in mathematical notation:

$$\Delta y_1 + \Delta y_2 + \Delta y_3 + \cdots + \Delta y_i + \cdots + \Delta y_n = 0 \quad (1.01)$$

$$\Delta x_1 + \Delta x_2 + \Delta x_3 + \cdots + \Delta x_i + \cdots + \Delta x_n = 0 \quad (1.02)$$

The latitudes and departures can be calculated, using bearings of the sides, but it is more efficient to employ azimuths. For these problems an azimuth is the direction of a line reckoned *clockwise* from North. If β is the "bearing" angle, the following formulae are employed to obtain the azimuth of the line:

$$\begin{array}{ll} N \beta E, & \alpha = \beta \\ S \beta E, & \alpha = 180^\circ - \beta \\ S \beta W, & \alpha = 180^\circ + \beta \\ N \beta W, & \alpha = 360^\circ - \beta \end{array}$$

The formulae to calculate the latitude and departure for line i are:

$$\text{Latitude: } \Delta y_i = s_i \cos \alpha_i \quad (1.03)$$

$$\text{Departure: } \Delta x_i = s_i \sin \alpha_i \quad (1.04)$$

Note: The sign of $\cos \alpha_i$ and $\sin \alpha_i$ is + or -, depending on the numerical value of the azimuth, α_i .

Substituting (1.03) into (1.01), and (1.04) into (1.02):

$$s_1 \cos \alpha_1 + s_2 \cos \alpha_2 + s_3 \cos \alpha_3 + \cdots + s_n \cos \alpha_n = 0 \quad (1.05)$$

$$s_1 \sin \alpha_1 + s_2 \sin \alpha_2 + s_3 \sin \alpha_3 + \cdots + s_n \sin \alpha_n = 0 \quad (1.06)$$

Or, rewriting in mathematical notation:

$$\sum_{i=1}^n s_i \cos \alpha_i = 0 \quad (1.07)$$

$$\sum_{i=1}^n s_i \sin \alpha_i = 0 \quad (1.08)$$

Now, anywhere in the polygon (and not necessarily adjacent sides) two elements (two distances, one distance and one azimuth, or two azimuths) are unknown, or omitted. For convenience call these sides 1 and 2. (For mathematical purposes we can "rearrange" the order of the terms in (1.05) and (1.06), and not alter the problem.) Rewriting (1.07) and (1.08):

$$s_1 \cos \alpha_1 + s_2 \cos \alpha_2 + \sum_{i=3}^n s_i \cos \alpha_i = 0 \quad (1.09)$$

$$s_1 \sin \alpha_1 + s_2 \sin \alpha_2 + \sum_{i=3}^n s_i \sin \alpha_i = 0 \quad (1.10)$$

Equations (1.09) and (1.10) are the same as (1.07) and (1.08), respectively. The third term in each equation is the sum of the latitudes and the sum of the departures for all the sides whose azimuth and distance are known. Since this is a number, let:

$$A = \sum_{i=3}^n s_i \sin \alpha_i \quad (1.11)$$

$$B = \sum_{i=3}^n s_i \cos \alpha_i \quad (1.12)$$

Then, (1.09) and (1.10), can be rewritten:

$$s_1 \cos \alpha_1 + s_2 \cos \alpha_2 + B = 0 \quad (1.13)$$

$$s_1 \sin \alpha_1 + s_2 \sin \alpha_2 + A = 0 \quad (1.14)$$

Equations (1.13) and (1.14) are the general

condition equations; which must be satisfied, when two of the four quantities (s_1 , s_2 , α_1 , and α_2) are unknown.

Case I: Two Distances Are Unknown

[s_1 and s_2].
[Formulae 2.01-2.04]

Multiply (1.14) by $\cos \alpha_1$, and multiply (1.13) by $-\sin \alpha_1$:

$$\begin{aligned} -s_1 \sin \alpha_1 \cos \alpha_1 - s_2 \sin \alpha_1 \cos \alpha_2 - B \sin \alpha_1 &= 0 \\ s_1 \sin \alpha_1 \cos \alpha_1 + s_2 \sin \alpha_2 \cos \alpha_1 + A \cos \alpha_1 &= 0 \end{aligned}$$

Add the resulting equations together and collect all terms that do not contain s_2 on the right-hand side of the equation.

$$s_2 [\sin \alpha_2 \cos \alpha_1 - \cos \alpha_2 \sin \alpha_1] = B \sin \alpha_1 - A \cos \alpha_1$$

Solve for s_2 :

$$s_2 = \frac{B \sin \alpha_1 - A \cos \alpha_1}{\sin \alpha_2 \cos \alpha_1 - \cos \alpha_2 \sin \alpha_1} \quad (2.01)$$

Recall:

$$\sin [x - y] = \sin x \cos y - \cos x \sin y$$

Then, (2.01) can be written:

$$s_2 = \frac{B \sin \alpha_1 - A \cos \alpha_1}{\sin [\alpha_2 - \alpha_1]} \quad (2.02)$$

Now, multiply (1.13) by $-\sin \alpha_2$, and multiply (1.14) by $\cos \alpha_2$:

$$\begin{aligned} -s_1 \sin \alpha_2 \cos \alpha_1 - s_2 \sin \alpha_2 \cos \alpha_2 - B \sin \alpha_2 &= 0 \\ s_1 \sin \alpha_1 \cos \alpha_2 + s_2 \sin \alpha_2 \cos \alpha_2 + A \cos \alpha_2 &= 0 \end{aligned}$$

Add the resulting equations together and collect all terms that do not contain s_1 on the right-hand side of the equation.

$$s_1 [\sin \alpha_1 \cos \alpha_2 - \sin \alpha_2 \cos \alpha_1] = B \sin \alpha_2 - A \cos \alpha_2$$

Solve for s_1 :

$$s_1 = \frac{B \sin \alpha_2 - A \cos \alpha_2}{\sin \alpha_1 \cos \alpha_2 - \sin \alpha_2 \cos \alpha_1}$$

Simplify:

$$s_1 = \frac{B \sin \alpha_2 - A \cos \alpha_2}{\sin [\alpha_1 - \alpha_2]} \quad (2.03)$$

Multiply the denominator of (2.02) by -1 :

$$-\sin [\alpha_2 - \alpha_1] = -\sin \alpha_2 \cos \alpha_1 + \cos \alpha_2 \sin \alpha_1$$

But:

$$\sin [\alpha_1 - \alpha_2] = \sin \alpha_1 \cos \alpha_2 - \cos \alpha_1 \sin \alpha_2$$

Therefore, (2.03) can be rewritten:

$$s_1 = \frac{B \sin \alpha_2 - A \cos \alpha_2}{-\sin [\alpha_2 - \alpha_1]} \quad (2.04)$$

Case II: One Distance and One Azimuth Are Unknown

[s_1 and α_2].
[Formulae 3.01-3.11]

Multiply (1.13) by $-\sin \alpha_1$, and multiply (1.14) by $\cos \alpha_1$:

$$\begin{aligned} -s_1 \sin \alpha_1 \cos \alpha_1 - s_2 \cos \alpha_2 \sin \alpha_1 - B \sin \alpha_1 &= 0 \\ s_1 \sin \alpha_1 \cos \alpha_1 + s_2 \sin \alpha_2 \cos \alpha_1 + A \cos \alpha_1 &= 0 \end{aligned}$$

Add the resulting equations together and collect all terms that do not contain α_2 on the right-hand side of the equation.

$$s_2 [\sin \alpha_2 \cos \alpha_1 - \cos \alpha_2 \sin \alpha_1] = B \sin \alpha_1 - A \cos \alpha_1$$

Divide both sides by $s_2 \cos \alpha_1$ [recall. $\sin x/\cos x = \tan x$]:

$$\sin \alpha_2 - \tan \alpha_1 \cos \alpha_2 = \frac{B \sin \alpha_1 - A \cos \alpha_1}{s_2 \cos \alpha_1} \quad (3.01)$$

Let:

$$\nu = \tan \alpha_1 \quad (3.02)$$

$$U = \frac{B \sin \alpha_1 - A \cos \alpha_1}{s_2 \cos \alpha_1} \quad (3.03)$$

Substitute these into (3.01):

$$\sin \alpha_2 - \nu \cos \alpha_2 = U$$

Or:

$$\sin \alpha_2 = U + \nu \cos \alpha_2 \quad (3.04)$$

Recall:

$$\sin \alpha_2 = \sqrt{1 - \cos^2 \alpha_2}$$

Then, (3.04) is:

$$\sqrt{1 - \cos^2 \alpha_2} = U + \nu \cos \alpha_2$$

Square both sides:

$$1 - \cos^2 \alpha_2 = U^2 + 2U\nu \cos \alpha_2 + \nu^2 \cos^2 \alpha_2$$

Rearranging:

$$\cos^2 \alpha_2 + \nu^2 \cos^2 \alpha_2 + 2U\nu \cos \alpha_2 + U^2 - 1 = 0$$

Or:

$$[1 + \nu^2] \cos^2 \alpha_2 + 2U\nu \cos \alpha_2 + [U^2 - 1] = 0$$

Employing the quadratic formula and solving for $\cos \alpha_2$:

$$\cos \alpha_2 = \frac{-2U\nu \pm \sqrt{4U^2\nu^2 - 4(1 + \nu^2)(U^2 - 1)}}{2(1 + \nu^2)}$$

Factor out $\sqrt{4}$. Then:

$$\cos \alpha_2 = \frac{-U\nu \pm \sqrt{U^2\nu^2 - [U^2 + U^2\nu^2 - 1 - \nu^2]}}{(1 + \nu^2)}$$

Or:

$$\cos \alpha_2 = \frac{-U\nu \pm \sqrt{\nu^2 + 1 - U^2}}{(1 + \nu^2)} \quad (3.05)$$

Solve (1.13) and (1.14) for $\cos \alpha_2$ and $\sin \alpha_2$, respectively:

$$\cos \alpha_2 = -\frac{s_1}{s_2} \cos \alpha_1 - \frac{B}{s_2} \quad (3.06)$$

$$\sin \alpha_2 = -\frac{s_1}{s_2} \sin \alpha_1 - \frac{A}{s_2} \quad (3.07)$$

Recall:

$$\cos^2 \alpha_2 = 1 - \sin^2 \alpha_2$$

Substitute (3.06) and (3.07) into this equation:

$$\frac{s_1^2}{s_2^2} \cos^2 \alpha_1 + \frac{2s_1 B}{s_2^2} \cos \alpha_1 + \frac{B^2}{s_2^2} = 1 - \frac{s_1^2}{s_2^2} \sin^2 \alpha_1 - \frac{2s_1 A}{s_2^2} \sin \alpha_1 - \frac{A^2}{s_2^2}$$

Clear the equation of fractions and collect terms:

$$(\cos^2 \alpha_1 + \sin^2 \alpha_1) s_1^2 + (2B \cos \alpha_1 + 2A \sin \alpha_1) s_1 + (B^2 + A^2 - s_2^2) = 0$$

Then:

$$s_1^2 + 2(A \sin \alpha_1 + B \cos \alpha_1) s_1 + (A^2 + B^2 - s_2^2) = 0 \quad (3.08)$$

Let:

$$U = A^2 + B^2 - s_2^2 \quad (3.09)$$

$$V = A \sin \alpha_1 + B \cos \alpha_1 \quad (3.10)$$

Then, (3.08) can be written:

$$s_1^2 + 2Vs_1 + U = 0$$

Employing the quadratic formula and solving for s_1 :

$$s_1 = \frac{-2V \pm \sqrt{4V^2 - 4U}}{2} \\ s_1 = -V \pm \sqrt{V^2 - U} \quad (3.11)$$

Note: When solving for α_2 and s_1 , there are four solutions for α_2 and two solutions for s_1 . Normally, two of the solutions for α_2 can be rejected immediately, and usually one of the values of s_1 can be rejected. If the problem is drawn to scale, it will be easy to obtain the correct solutions. However, if the two "acceptable" values of α_2 are nearly the same, insert the values into (1.13) and (1.14). Only one of these values will satisfy the equation.

Case II: Unknown Distance and Azimuth Are for the Same Line

[s_1 and α_1].

[Formulae 4.01-4.06]

Let:

$$D = B + s_2 \cos \alpha_2 \quad (4.01)$$

$$C = A + s_2 \sin \alpha_2 \quad (4.02)$$

Then, (1.13) and (1.14) can be written:

$$s_1 \cos \alpha_1 = -D \quad (4.03)$$

$$s_1 \sin \alpha_1 = -C \quad (4.04)$$

Divide (4.04) by (4.03):

$$\tan \alpha_1 = \frac{s_1 \sin \alpha_1}{s_1 \cos \alpha_1} = \frac{C}{D} \quad (4.05)$$

Square (4.03) and (4.04) and add together:

$$s_1^2 \cos^2 \alpha_1 + s_1^2 \sin^2 \alpha_1 = D^2 + C^2$$

Or:

$$(\cos^2 \alpha_1 + \sin^2 \alpha_1) s_1^2 = D^2 + C^2$$

Or:

$$s_1^2 = C^2 + D^2$$

Then:

$$s_1 = \sqrt{D^2 + C^2} \quad (4.06)$$

Case III: Two Azimuths Are Unknown

[α_1 and α_2].

[Formulae 5.01-5.12]

Solve (1.14) for $\sin \alpha_1$:

$$\sin \alpha_1 = -\frac{A}{s_1} - \frac{s_2}{s_1} \sin \alpha_2 \quad (5.01)$$

Recall:

$$\cos^2 \alpha_1 = 1 - \sin^2 \alpha_1$$

Substitute (5.01) into this equation:

$$\cos^2 \alpha_1 = 1 - \left[\frac{A^2}{s_1^2} + \frac{2As_2}{s_1^2} \sin \alpha_2 + \frac{s_2^2}{s_1^2} \sin^2 \alpha_2 \right] \\ = \frac{s_1^2 - A^2}{s_1^2} - \frac{2As_2}{s_1^2} \sin \alpha_2 - \frac{s_2^2}{s_1^2} \sin^2 \alpha_2$$

Therefore:

$$\cos \alpha_1 = \frac{1}{s_1} \sqrt{s_1^2 - A^2 - 2As_2 \sin \alpha_2 - s_2^2 \sin^2 \alpha_2}$$

Substitute this expression for $\cos \alpha_1$ into (1.13):

$$\sqrt{s_1^2 - A^2 - 2As_2 \sin \alpha_2 - s_2^2 \sin^2 \alpha_2} + s_2 \cos \alpha_2 + B = 0$$

Transfer the second and third terms of this expression to the right-hand side of the equation and square both sides:

$$s_1^2 - A^2 - 2As_2 \sin \alpha_2 - s_2^2 \sin^2 \alpha_2 = B^2 + 2Bs_2 \cos \alpha_2 + s_2^2 \cos^2 \alpha_2$$

Rearranging:

$$s_1^2 - A^2 - 2As_2 \sin \alpha_2 - (\sin^2 \alpha_2 + \cos^2 \alpha_2) s_2^2 - B^2 - 2Bs_2 \cos \alpha_2 = 0 \quad (5.02)$$

Recall:

$$\sin^2 \alpha_2 + \cos^2 \alpha_2 = 1$$

Therefore, (5.02) becomes:

$$s_1^2 - A^2 - B^2 - s_2^2 - 2As_2 \sin \alpha_2 - 2Bs_2 \cos \alpha_2 = 0$$

Multiply by -1, and solve for $\sin \alpha_2$.

$$\sin \alpha_2 = -\frac{2Bs_2}{2As_2} \cos \alpha_2 - \frac{1}{2As_2} (A^2 + B^2 - s_1^2 + s_2^2) \quad (5.03)$$

Let:

$$U = \frac{1}{2As_2} (A^2 + B^2 - s_1^2 + s_2^2) \quad (5.04)$$

Then, (5.03) becomes:

$$\sin \alpha_2 = -\frac{B}{A} \cos \alpha_2 - U \quad (5.05)$$

Recall:

$$\sin \alpha_2 = \sqrt{1 - \cos^2 \alpha_2}$$

Substitute this expression into (5.05)

$$\sqrt{1 - \cos^2 \alpha_2} = -\frac{B}{A} \cos \alpha_2 - U$$

Square both sides:

$$1 - \cos^2 \alpha_2 = \frac{B^2}{A^2} \cos^2 \alpha_2 + \frac{2BU}{A} \cos \alpha_2 + U^2$$

Rearranging:

$$\frac{B^2 + A^2}{A^2} \cos^2 \alpha_2 + \frac{2BU}{A} \cos \alpha_2 + (U^2 - 1) = 0$$

Employing the quadratic formula, and solving for $\cos \alpha_2$:

$$\cos \alpha_2 = \frac{-\frac{2BU}{A} \pm \sqrt{\frac{4B^2U^2}{A^2} - 4\left(\frac{B^2 + A^2}{A^2}\right)(U^2 - 1)}}{2\frac{B^2 + A^2}{A^2}}$$

Simplifying:

$$\cos \alpha_2 = \frac{-BU \pm \sqrt{A^2 + B^2 - A^2U^2}}{\frac{B^2 + A^2}{A}} \quad (5.06)$$

Solve (1.14) for $\sin \alpha_2$:

$$\sin \alpha_2 = -\frac{A}{s_2} - \frac{s_1}{s_2} \sin \alpha_1 \quad (5.07)$$

Then, following the same reasoning:

$$\begin{aligned} \cos^2 \alpha_2 &= 1 - \left[\frac{A^2}{s_2^2} + \frac{2As_1}{s_2^2} \sin \alpha_1 + \frac{s_1^2}{s_2^2} \sin^2 \alpha_1 \right] \\ &= \frac{s_2 - A^2}{s_2^2} - \frac{2As_1}{s_2^2} \sin \alpha_1 - \frac{s_1^2}{s_2^2} \sin^2 \alpha_1 \end{aligned}$$

Then:

$$\cos \alpha_2 = \frac{1}{s_2} \sqrt{s_2^2 - A^2 - 2As_1 \sin \alpha_1 - s_1^2 \sin^2 \alpha_1}$$

Substitute this expression for $\cos \alpha_2$ in (1.13):

$$\sqrt{s_2^2 - A^2 - 2As_1 \sin \alpha_1 - s_1^2 \sin^2 \alpha_1} + s_1 \cos \alpha_1 + B = 0$$

Transfer the second and third terms of this expression to the right-hand side of the equation and square both sides:

$$s_2^2 - A^2 - 2As_1 \sin \alpha_1 - s_1^2 \sin^2 \alpha_1 = B^2 + 2Bs_1 \cos \alpha_1 + s_1^2 \cos^2 \alpha_1$$

Rearranging:

$$s_2^2 - A^2 - 2As_1 \sin \alpha_1 - (\sin^2 \alpha_1 + \cos^2 \alpha_1) s_1^2 - B^2 - 2Bs_1 \cos \alpha_1 = 0 \quad (5.08)$$

Then:

$$s_2^2 - A^2 - B^2 - s_1^2 - 2As_1 \sin \alpha_1 - 2Bs_1 \cos \alpha_1 = 0$$

Multiply by -1 and solve for $\sin \alpha_1$:

$$\sin \alpha_1 = -\frac{2Bs_1}{2As_1} \cos \alpha_1 - \frac{1}{2As_1} (A^2 + B^2 - s_2^2 + s_1^2) \quad (5.09)$$

Let:

$$U = \frac{1}{2As_1} (A^2 + B^2 - s_2^2 + s_1^2) \quad (5.10)$$

Then, (5.09) becomes:

$$\sin \alpha_1 = -\frac{B}{A} \cos \alpha_1 - U \quad (5.11)$$

Then:

$$\sqrt{1 - \cos^2 \alpha_1} = -\frac{B}{A} \cos \alpha_1 - U$$

And:

$$1 - \cos^2 \alpha_1 = \frac{B^2}{A^2} \cos^2 \alpha_1 + \frac{2BU}{A} \cos \alpha_1 + U^2$$

Rearranging:

$$\frac{B^2 + A^2}{A^2} \cos^2 \alpha_1 + \frac{2BU}{A} \cos \alpha_1 + (U^2 - 1) = 0$$

Employing the quadratic formula, and solving for $\cos \alpha_2$:

$$\cos \alpha_1 = \frac{-\frac{2BU}{A} \pm \sqrt{\frac{4B^2U^2}{A^2} - 4\left(\frac{B^2 + A^2}{A^2}\right)(U^2 - 1)}}{2\frac{B^2 + A^2}{A^2}}$$

Simplifying:

$$\cos \alpha_1 = \frac{-BU \pm \sqrt{A^2 + B^2 - A^2U^2}}{\frac{B^2 + A^2}{A}} \quad (5.12)$$

Note: When solving (5.06) and (5.12), there are eight possible solutions. However, four

solutions usually can be rejected immediately. If the two lines are not nearly parallel and if the polygon has been drawn to scale, then it is relatively simple to select the correct solution.

Additional Comments

The formulae derived will solve any closed polygon with two unknown elements. However, if the angle between the two lines with the unknown elements is small (less than about 20°) the solution may be weak, and the round-off error may be significant.

Numerical Examples

Figure 1 will be employed to illustrate the three problems. Known data for the three examples:

<i>i</i>	α_i	sine	cosine	s_i
1	36° 42' 25"	+0.5977 2232	+0.8017 0320	468.38
3	193° 02' 56"	-0.2257 8237	-0.9741 7777	723.00
5	346° 28' 20"	-0.2339 1675	+0.9722 5663	967.30

For all numerical examples [formulae (1.11) and (1.12)]:

$$A = s_1 \sin \alpha_1 + s_3 \sin \alpha_3 + s_5 \sin \alpha_5 = -109.54715$$

$$B = s_1 \cos \alpha_1 + s_3 \cos \alpha_3 + s_5 \cos \alpha_5 = +611.63506$$

Case Ia:
[Given: α_2 and α_4 ; compute s_2 and s_4].

<i>i</i>	α_i	sine	cosine
2	97° 34' 01"	+0.9912 9168	-0.1316 8451
4	222° 15' 08"	-0.6723 9552	-0.7401 9205

Compute s_4 from formula (2.02) (with the correct subscripts for this problem).

$$(\alpha_4 - \alpha_2) = 124^\circ 41' 07''$$

$$\sin(\alpha_4 - \alpha_2) = +0.8222 \ 9029$$

$$s_4 = \frac{B \sin \alpha_2 - A \cos \alpha_2}{\sin [\alpha_4 - \alpha_2]}$$

$$= 719.798 \ (719.80)$$

Compute s_2 from formula (2.04).

$$s_2 = \frac{B \sin \alpha_4 - A \cos \alpha_4}{-\sin [\alpha_4 - \alpha_2]}$$

$$= 598.750 \ (598.75)$$

Case IIb:
[Given: α_2 and s_4 ; compute α_4 and s_2].

<i>i</i>	α_i	sine	cosine
2	97° 34' 01"	+0.9912 9168	-0.1316 8451

$s_4 = 719.80$

Calculate ν and U from (3.02) and (3.03) respectively [with correct subscripts for this problem].

$$\nu = \tan \alpha_2$$

$$= -7.5277 \ 7756$$

$$U = \frac{B \sin \alpha_2 - A \cos \alpha_2}{s_4 \cos \alpha_2}$$

$$= -6.2443 \ 8099$$

Insert these values into formula (3.05).

$$\cos \alpha_4 = \frac{-U\nu \pm \sqrt{\nu^2 + 1 - U^2}}{(1 + \nu^2)}$$

$$= \frac{-47.0063 \ 1109 \pm 4.3214 \ 7414}{57.6674 \ 3499}$$

$$= -0.7401 \ 8962 \ \left\{ \begin{array}{l} 137^\circ 44' 51''.3 \\ 222^\circ 15' 08''.7 \end{array} \right.$$

$$= -0.8900 \ 6534 \ \left\{ \begin{array}{l} 152^\circ 52' 53''.3 \\ 207^\circ 07' 06''.7 \end{array} \right.$$

Reject 137° 44' 51".3 and 152° 52' 53".3 (the azimuth of line 4 does not match these values).

Calculate V and U from (3.09) and (3.10), respectively (with correct subscripts for this problem).

$$U = A^2 + B^2 - s_4^2$$

$$= -132,014.015$$

$$V = A \sin \alpha_2 + B \cos \alpha_2$$

$$= -189.13604$$

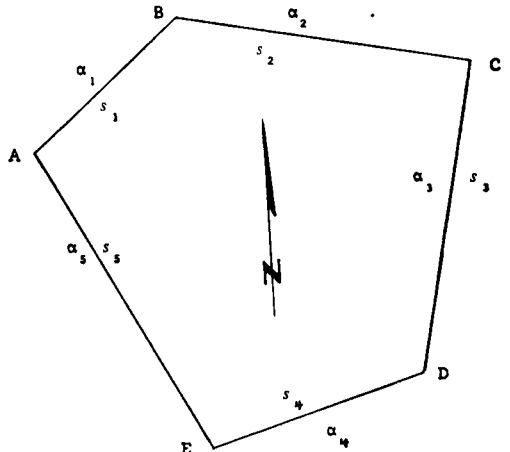


Figure 1.

Insert these values into (3.11).

$$s_2 = -V \pm \sqrt{V^2 - U}$$

$$= +189.1360 \pm 409.6174$$

$$= +598.753 \quad (598.75)$$

$$= -220.481$$

Reject -220.481 (distance cannot be negative). To select the correct value of α_4 , insert both values of α_4 into formula (1.14).

$$\alpha_4 = 222^\circ 15' 08'' .7 \quad (\text{sine} = -0.6723 \quad 9803) + 593.539$$

$$- 483.992 - 109.547 = 0.000$$

$$\alpha_4 = 207^\circ 07' 06'' .7 \quad (\text{sine} = -0.4558 \quad 3275) + 593.539$$

$$- 328.108 - 109.547 = +155.884$$

Therefore, $\alpha_4 = 222^\circ 15' 08'' .7$ is the correct azimuth.

Case IIIc:

[Given s_2 and s_4 ; compute α_2 and α_4]

$$s_2 = 598.75 \quad \text{and} \quad s_4 = 719.80$$

$$U = \frac{1}{2As_2} (A^2 + B^2 - s_4^2 + s_2^2)$$

$$= -1.7265 \quad 0453$$

Insert these values into (5.12).

$$\cos \alpha_2 = \frac{-BU \pm \sqrt{A^2 + B^2 - A^2U^2}}{A^2 + B^2}$$

$$= \frac{+1,055.99070 \pm 591.88384}{-3,524.4917}$$

$$= -0.4675 \quad 4956 \quad 117^\circ 52' 31'' .3$$

$$= -0.1316 \quad 8051 \quad 242^\circ 07' 28'' .7$$

$$= -0.1316 \quad 8051 \quad 97^\circ 34' 00'' .2$$

$$= -0.1316 \quad 8051 \quad 262^\circ 25' 59'' .8$$

From the figure, the values $242^\circ 07' 28'' .7$ and $262^\circ 25' 59'' .8$ can be rejected.

Now, obtain the sines and cosines for the values of α_2 and α_4 that remain.

No.	i	α_i	sine	cosine
1	2	117° 52' 31" .3	+0.8839 6677	-0.4675 4972
2	2	97° 34' 00" .2	+0.9912 9219	-0.1316 8066
3	4	222° 15' 07" .5	-0.6723 9372	-0.7401 9368
4	4	242° 33' 38" .6	-0.8874 9970	-0.4608 0830

Calculate U from (5.04) (with correct subscripts for this problem).

$$U = \frac{1}{2As_4} (A^2 + B^2 - s_2^2 + s_4^2)$$

$$= -3.4603 \quad 3224$$

$$\frac{A^2 + B^2}{A} = -3,524.4917$$

Insert these values into (5.06).

$$\cos \alpha_4 = \frac{-BU \pm \sqrt{A^2 + B^2 - A^2U^2}}{A^2 + B^2}$$

$$= \frac{+2,116.46052 \pm 492.34572}{-3,524.4917}$$

$$= -0.7401 \quad 9361 \quad 137^\circ 44' 52'' .5$$

$$= -0.4608 \quad 0823 \quad 222^\circ 15' 07'' .5$$

$$= -0.4608 \quad 0823 \quad 117^\circ 26' 21'' .4$$

$$= -0.4608 \quad 0823 \quad 242^\circ 33' 38'' .6$$

From the figure, the values $137^\circ 44' 52'' .5$ and $117^\circ 26' 21'' .4$ can be rejected.

Calculate U from (5.10) [with correct subscripts for this problem].

Insert these values in various combinations into formulae (1.13) and (1.14) to obtain the correct azimuths. For (1.14) the answers for the combinations 1 and 3, 1 and 4, 2 and 3, and 2 and 4 respectively are: -64.261, -219.094, 0.000, and -154.833. The correct answers are:

$$\alpha_2 = 97^\circ 34' 00'' .2$$

$$\alpha_4 = 222^\circ 15' 07'' .5$$

Acknowledgments

The author would like to thank Paul W. Schreckengost, a senior in Surveying Technology, for his work on proofreading the manuscript and verifying the numerical examples.

Reference

Johnson, John Butler, *The Theory and Practice of Surveying*, 16th edition; John Wiley & Son; New York; pp. 211-215. ■