Circle Fitting

L.B. CAPON

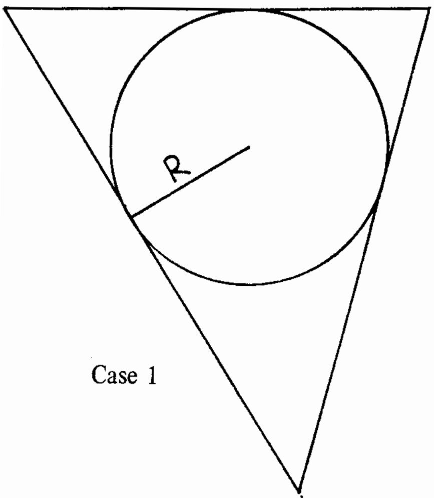
Abstract

With the emphasis on road safety, "roundabouts" are being constructed at major intersections. Although roundabouts may be of various shapes, this paper looks at circle roundabouts and the mathematics of determining the radius of a circle which is tangential to straights and curves.

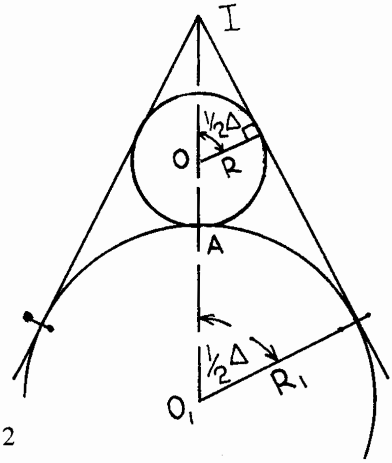
Introduction

The problem of fitting a circle tangent to straights and/or curves can be often reduced to four cases.

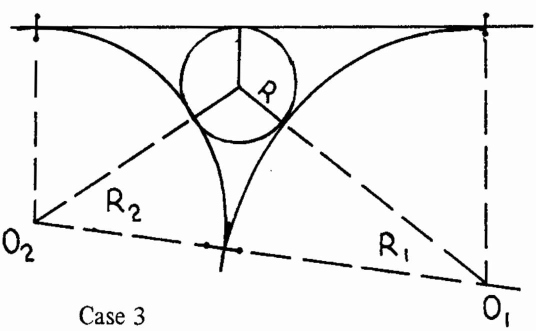
1. Tangential to 3 straights
2. Tangential to 2 straights and 1 curve
3. Tangential to 1 straight and 2 curves
4. Tangential to 3 curves



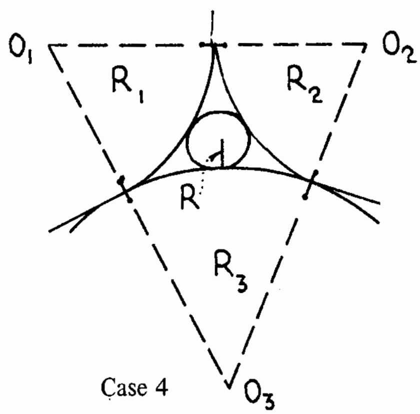
Case 1



Case 2



Case 3



Case 4

L.B. CAPON, L.S., M.I.S. Aust., F.I.E.S.Q., Head of Surveying, School of Engineering, D.D.I.A.E.

A complicated intersection consisting of straights and curves may generally be reduced to any one of these four cases. However, which one may be difficult to detect unless a good quality scale plot is drawn of the data. A transparency of concentric circles can then be used to select the best scaled circle and determine which of the four cases is required.

It is not necessary that the curves have common tangent points as drawn above.

Case 1 This is the elementary geometrical case of an inscribed circle in a triangle where the bisectors of the angles of the triangle intersect at the centre.

Case 2 This is also an elementary problem. Now when Δ is the deflection angle:

$$\cos \tfrac{1}{2}\Delta = \frac{R}{OI} = \frac{R_1}{O_1 I}$$

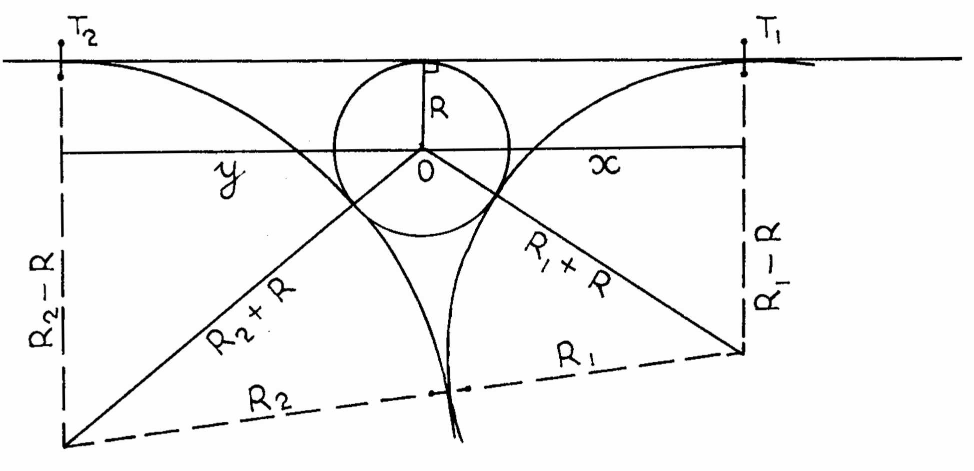
$$IO + R + R_1 = O_1 I$$

$$\frac{R}{\cos \tfrac{1}{2}\Delta} + R + R_1 = \frac{R_1}{\cos \tfrac{1}{2}\Delta}$$

$$R\left(\frac{1}{\cos \tfrac{1}{2}\Delta} + 1\right) = R_1\left(\frac{1}{\cos \tfrac{1}{2}\Delta} - 1\right)$$

$$R = \frac{R_1\left(\frac{1}{\cos \tfrac{1}{2}\Delta} - 1\right)}{\left(\frac{1}{\cos \tfrac{1}{2}\Delta} + 1\right)} = R_1 \tan^2 \tfrac{1}{4}\Delta$$

Case 3 This does not need to be a common tangent.



Curves of radius R_1 and R_2 are fixed and the distance T_1T_2 (= ℓ) is known, i.e. $x + y = \ell$

Then

$$x^2 = (R_1 + R)^2 - (R_1 - R)^2$$

$$= R_1^2 + 2R_1R + R^2 - R_1^2 + 2R_1R - R^2$$

$$= 4R_1R$$

$$x = 2\sqrt{R_1R}$$

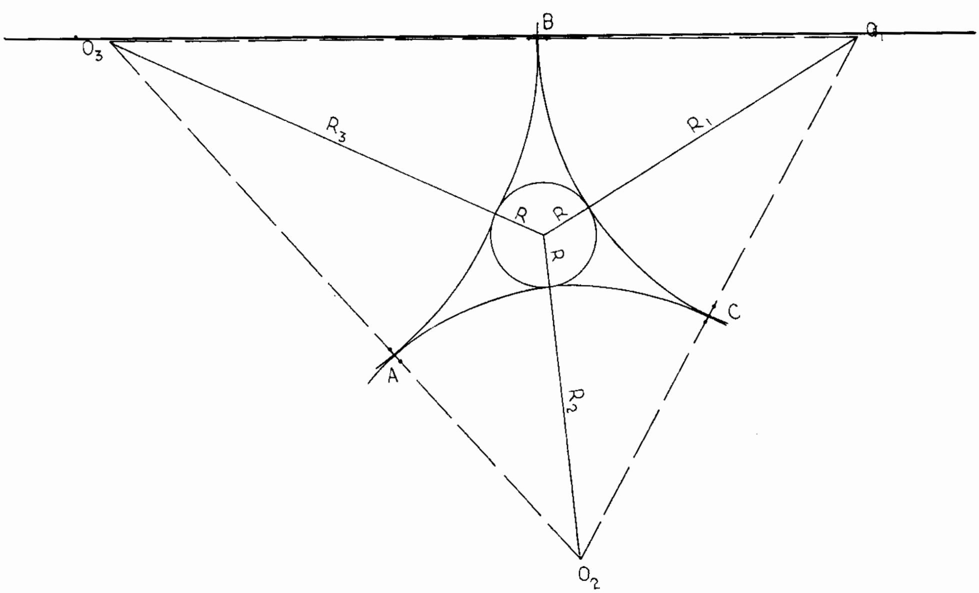
Similarly $y = 2\sqrt{R_2R}$

and $\ell = 2\sqrt{R_1}\sqrt{R} + 2\sqrt{R_2}\sqrt{R}$

$$\sqrt{R} = \frac{\ell}{2(\sqrt{R_1} + \sqrt{R_2})}$$

$$R = \left(\frac{\ell}{2(\sqrt{R_1} + \sqrt{R_2})}\right)^2$$

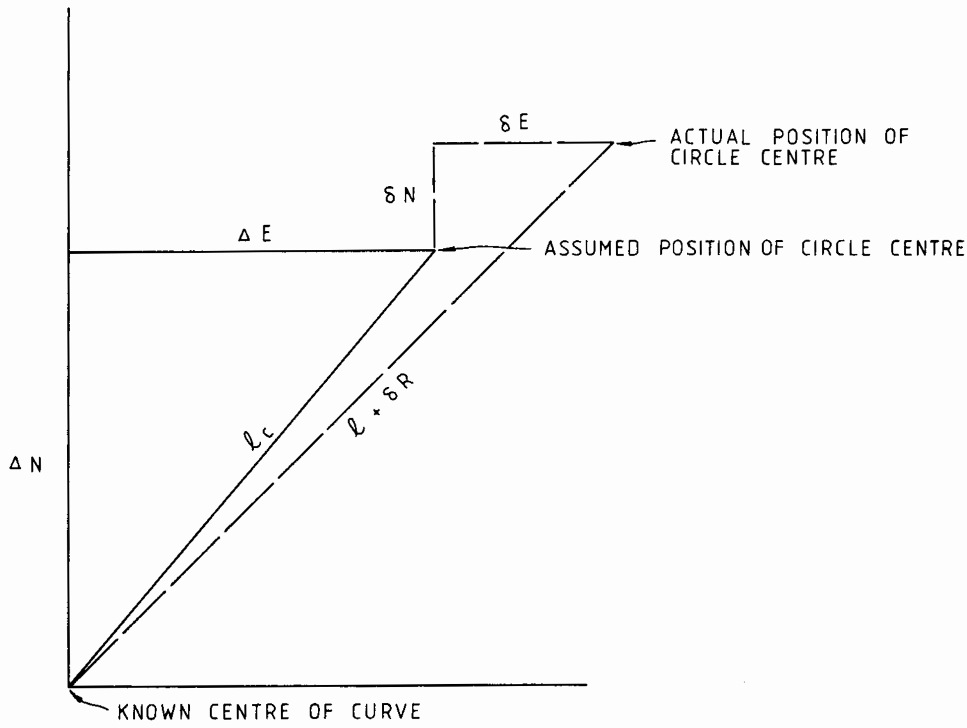
Case 4 A, B and C do not have to be common tangent points.



Let the three curves have centres at 0_1, 0_2 and 0_3 of radii R_1 R_2 and R_3
The required circle is of radius R centre 0.

Consider that the three given radii are unequal then 0 will not coincide with the intersection of any two tangent straights. In each curve, the line from its centre to 0 will not bisect the angle at the centre.
Thus, a direct solution is awkward. An elegant solution to the problem can be achieved by variation of co-ordinates.

Assume co-ordinates for 0 and a radius R for the circle and let δE, δN and δR be corrections to be applied to these assumed values. It then becomes a simple distance intersection problem with three distances being
$R_1 + R + \delta R$, $R_2, + R + \delta R$ and $R_3 + R + \delta R$ from the three known centres forming a 3 x 3 matrix.



Let ℓ be the known radius of the curve plus the assumed radius R.

Then

$$\ell_c^2 = \Delta E^2 + \Delta N^2$$

$$(\ell + \delta R)^2 = (\Delta N + \delta N)^2 + (\Delta E + \delta E)^2$$

$$\ell^2 + 2\ell\delta R = \Delta N^2 + 2\Delta N\delta N + \Delta E^2 + 2\Delta E\delta E$$

$$\ell^2 - \ell^2 = 2\Delta N\delta N + 2\Delta E\delta E - 2\ell\delta R$$

$$\ell - \ell_C = \delta N \cos\Theta + \delta E \sin\Theta - \delta R$$

or re-arranged.

$$-\delta R + \delta E \sin\Theta + \delta N \cos\Theta = R_1 + R_A - \ell_C$$

where Θ is the computed bearing from the curve centre to the assumed co-ordinates of the circle centre. If the assumed values are greatly in error, iteration can overcome the problem. A programme for the HP41C using "prompts" and iteration to 0.0001m for δR took 324 Bytes and with END at 171.

Example Case 3

$R_1 = 150 \quad R_2 = 200 \quad$ and bearing T_2 to $T_1 = 35°$

By close	35°	( )	computes as 346.410
	125°	150	
	( )	350	computes as 206°47'12"
	305°	200	

$$\text{So} \quad \ell = 346.410$$

$$R = 2\left(\frac{346.410}{(\sqrt{150}+\sqrt{200})}\right)^2$$

$$= 43.078$$

$$x = 2\sqrt{R_1 R}$$

$$= 160.769$$

$$y = 2\sqrt{R_2 R}$$

$$= 185.641 \quad \text{and } x + y = 346.410 = \ell$$

Example Case 4

Station		Easting	Northing	Radius
0_1		500.000	10000.000	170
0_2		814.215	10060.571	150
0_3		748.229	9686.345	230
0	Assumed	685.000	9935.000	30

Forming the equations giving

			$-\delta R$	δE	δN	=
0_1	→	0	1	+0.94346	−0.33149	= + 3.91328
0_2	→	0	1	−0.71715	−0.69692	= − 0.17933
0_3	→	0	1	−0.24644	+0.96916	= + 3.43185

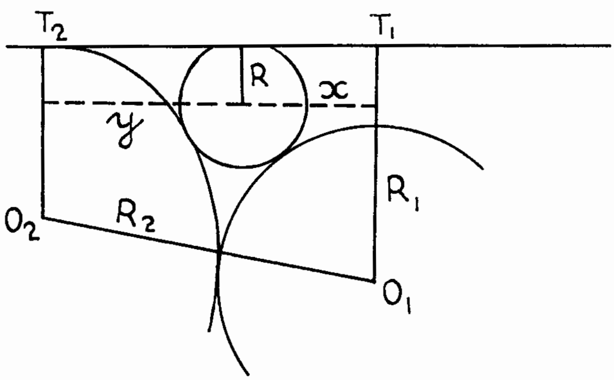
giving	δE	=	+2.1193	and	E	=	687.119
	δN	=	+1.5687		N	=	9936.569
	$-\delta R$	=	+2.4338		R	=	27.566

These corrections are too large and further iteration using the new assumed values gives:

δE	=	$-1.41 * 10^{-7}$	E	=	687.113
δN	=	$-4.39 * 10^{-7}$	N	=	9936.564
$-\delta R$	=	$-9.73 * 10^{-8}$	R	=	27.574

Exception

The above 4 cases can be adapted to suit varying requirements.



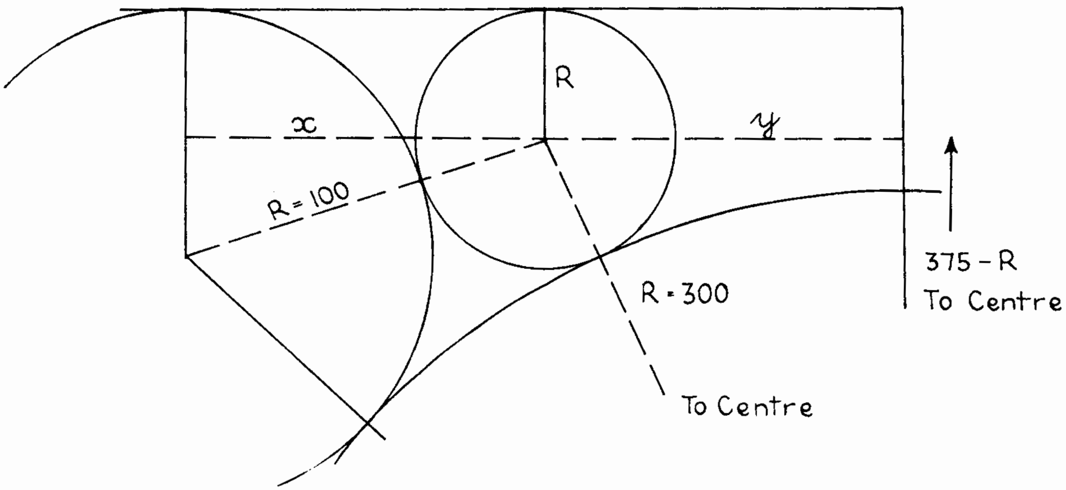
In this case if 0_1T is made perpendicular to T_2T it becomes a special application of case 3, where $0_1T - R$ is used instead of $R_1 - R$.

That is, $x^2 = (R_1 + R)^2 - (0_1T - R)^2$

Similarly, other adaptions can be made.

It will probably be easier in this case to firstly solve for x, thence for R rather than solve for R directly.

Example



$$x^2 = (100 + R)^2 - (100 - R)^2$$

which reduces to

$$R = \frac{x^2}{400}$$

$$y^2 = (300 + R)^2 - (375 - R)^2$$

$$R = \frac{y^2 + 50625}{1350} = \frac{x^2}{400}$$

$$x = 290.474 - y$$

$$x^2 = (290.474 - y)^2$$

$$\frac{y^2 + 50625}{1350} = \frac{290.474^2 + y^2 - 580.948\,y}{400}$$

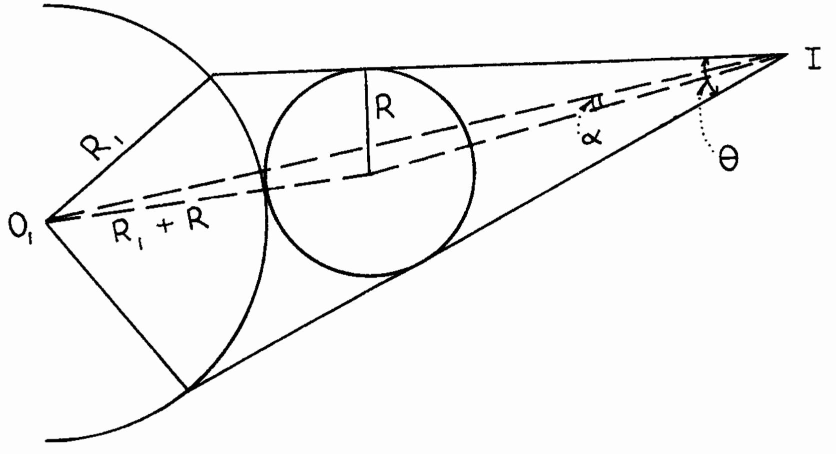
giving the quadratic equation as:

$$950\,y^2 - 784279.8y + 93656445.3 = 0$$

y = 680.735 or 144.822 (which is the required value) and by substitution
R = 53.036

Particular Case

One exception is for case 2 when the straights are not tangential to the curve.

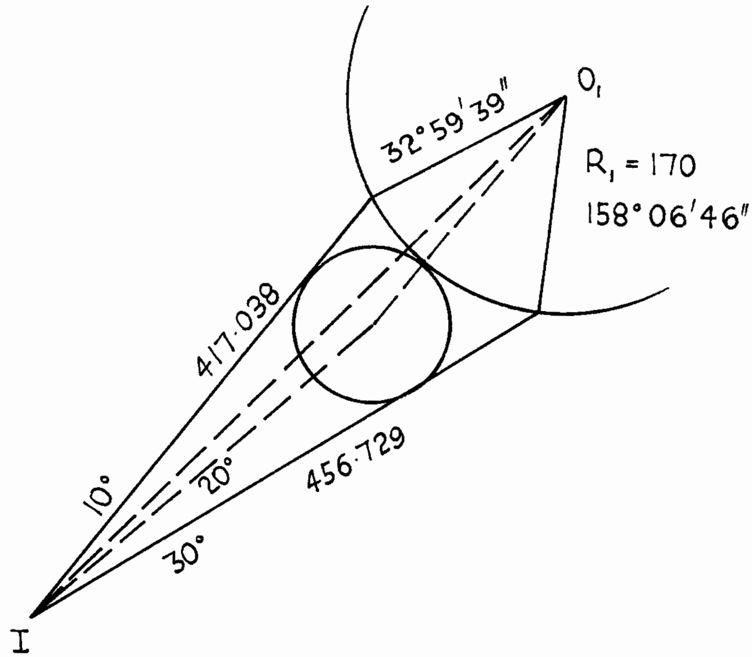


IO will lie on the bisector of the angle at I and so the angle OIO_1 can be determined.
OI = R/sin ½Θ, R can be solved from

$$\cos\alpha = \frac{O_1I^2 + (R/\sin \tfrac{1}{2}\Theta)^2 - (R_1 + R)^2}{2\,O_1I\,(R/\sin \tfrac{1}{2}\Theta)}$$

which is in the form of a quadratic equation. Two values for R will result. The smaller value will be the required radius.

Example



By close, $O_1 I$ = 16°36'17" 577.362

$O_1 IO$ = 3°23'43"

$$\cos 3°23'43" = \frac{577.362^2 + (R/\sin 10)^2 - (170 + R)^2}{2\,(R/\sin 10)\;577.362}$$

which reduces to

$$32.1634R^2 - 6978.1182R + 304446.879 = 0$$

from which R = 156.459 or 60.499. So required radius is 60.499.

Conclusion

It is hoped that this paper may assist those surveyors who are given the task of mathematically fitting circles to draftsmen's plotting and architects' dreams.

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