

AREA CUTOFF BY COORDINATES

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ABSTRACT

The method of calculating areas by coordinates is commonly used in land partitioning. Depending on the case under consideration, the coordinates of one or both endpoints of the cutoff line can be determined. The first case occurs when the cutoff line passes through a given point on the boundary, while the second case occurs when the cutoff line has a given bearing. In either case, equations equal in number to the unknown coordinates are found and solved simultaneously. As a result, formulae are found for the unknown coordinates as functions of the given data. Two numerical examples demonstrate the solutions of both cases.

INTRODUCTION

A common solution of land partitioning problems is to assume a cutoff line at an approximate location and then change its position, by trial-and-error procedures, until the desired condition is satisfied. Computer programs are helpful to speed up the iteration. The tracts of land, however, generally have particular shapes which requires the handling of every case separately. This leads to lengthy calculations. It is advantageous to have formulae in which data can be directly substituted to obtain the desired quantities. These formulae will also simplify computer programming. In a previous publication (Danial 1982), the author derived equations for the lengths of the cutoff and adjacent lines by using the Double-Meridian-Distance method. Using a similar approach, equations are derived here for the coordinates of one or both endpoints of the cutoff line as functions of the coordinates of the given points.

AREA BY COORDINATES

The area of a closed figure can be obtained by dividing the figure into trapezoids parallel to either the north axis or to the east axis. The first case is demonstrated in Fig. 1 where:

$$\begin{aligned}
 \text{Area } P_1 P_2 P_3 P_4 &= -(\text{area } P'_1 P_1 P_2 P'_2 + \text{area } P'_2 P_2 P_3 P'_3) \\
 &\quad + (\text{area } P'_3 P_3 P_4 P'_4 + \text{area } P'_4 P_4 P_1 P'_1) \\
 &= -\frac{1}{2}(N_1 + N_2)(E_2 - E_1) - \frac{1}{2}(N_2 + N_3)(E_3 - E_2) \\
 &\quad + \frac{1}{2}(N_3 + N_4)(E_3 - E_4) + \frac{1}{2}(N_4 + N_1)(E_4 - E_1) \quad (1)
 \end{aligned}$$

Exchanging the east coordinates in the first two negative terms and multiplying both sides of Eqn. 1 by two gives:

$$\begin{aligned}
 \text{Double area } P_1 P_2 P_3 P_4 &= (N_1 + N_2)(E_1 - E_2) + (N_2 + N_3)(E_2 - E_3) \\
 &\quad + (N_3 + N_4)(E_3 - E_4) + (N_4 + N_1)(E_4 - E_1) \quad (2)
 \end{aligned}$$

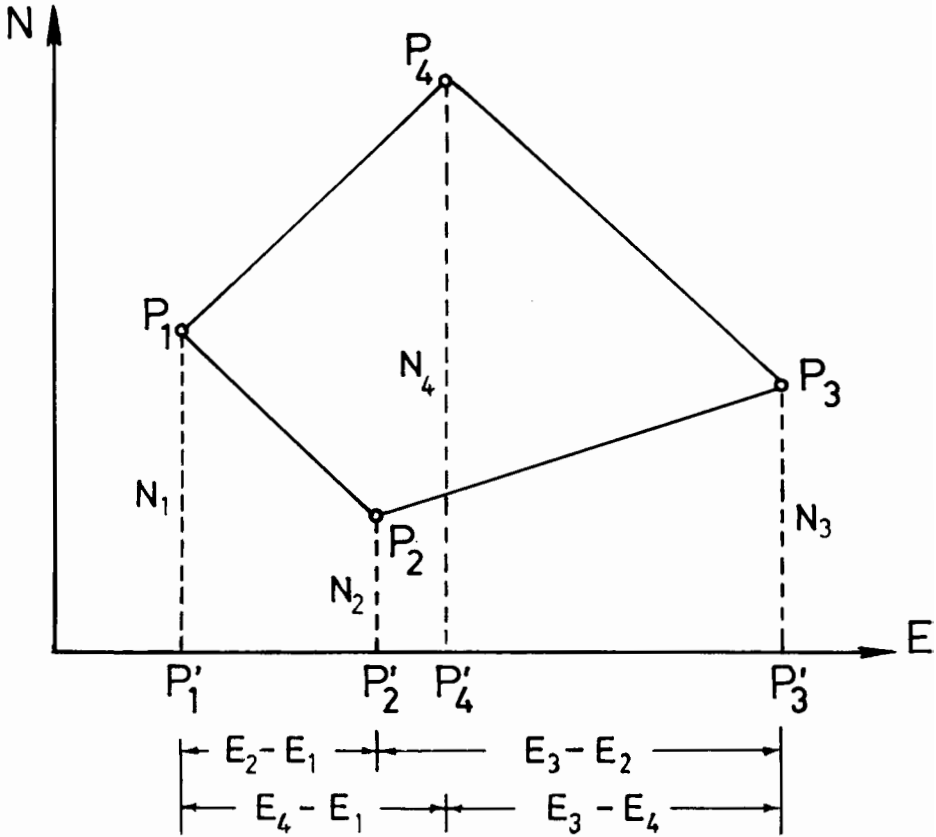


Fig. 1. Area $P_1P_2P_3P_4$ is obtained from trapezoids parallel to the north axis.

Eqn. 2 gives a positive value for the area $P_1P_2P_3P_4$ shown in Fig. 1. Had the corner points been taken in a clockwise direction, i.e. in the order $P_4, P_3, P_2,$ and P_1 , Eqn. 2 would have given a negative area. For area calculation the negative sign is neglected. In land partitioning, however, the area is introduced in the solution as a known quantity. Its algebraic sign is therefore significant.

Dividing the area in Fig. 1 into horizontal trapezoids parallel to the east axis will lead to a similar equation:

$$\begin{aligned} \text{Double area } P_1P_2P_3P_4 = & (N_1 - N_2)(E_1 + E_2) + (N_2 - N_3)(E_2 + E_3) \\ & + (N_3 - N_4)(E_3 + E_4) + (N_4 - N_1)(E_4 + E_1) \end{aligned} \quad (3)$$

For an n -sided traverse $P_1P_2P_3 \dots P_n$ the double area is:

$$\begin{aligned} = & \pm [(N_1 + N_2)(E_1 - E_2) + (N_2 + N_3)(E_2 - E_3) + \dots \\ & + (N_{n-1} + N_n)(E_{n-1} - E_n) + (N_n + N_1)(E_n - E_1)] \end{aligned} \quad (4)$$

or

$$= \pm [(N_1 - N_2)(E_1 + E_2) + (N_2 - N_3)(E_2 + E_3) + \dots + (N_{n-1} - N_n)(E_{n-1} + E_n) + (N_n - N_1)(E_n + E_1)] \quad (5)$$

The positive and negative signs in these patterns of calculations correspond to a counterclockwise and a clockwise order in which the corner points are taken. Usually both (4) and (5) are applied together to check the area calculation. The formulae derived in this paper are based on (4) only.

CASE 1: AREA CUTOFF FROM A POINT ON A BOUNDARY LINE

The problem can be summarized in the following way. From point P_1 on line QP_2 in Fig. 2, it is desired to create a new parcel $P_1P_2 \dots P_iP_{i+1} \dots P_{n-1}P_n$ of a

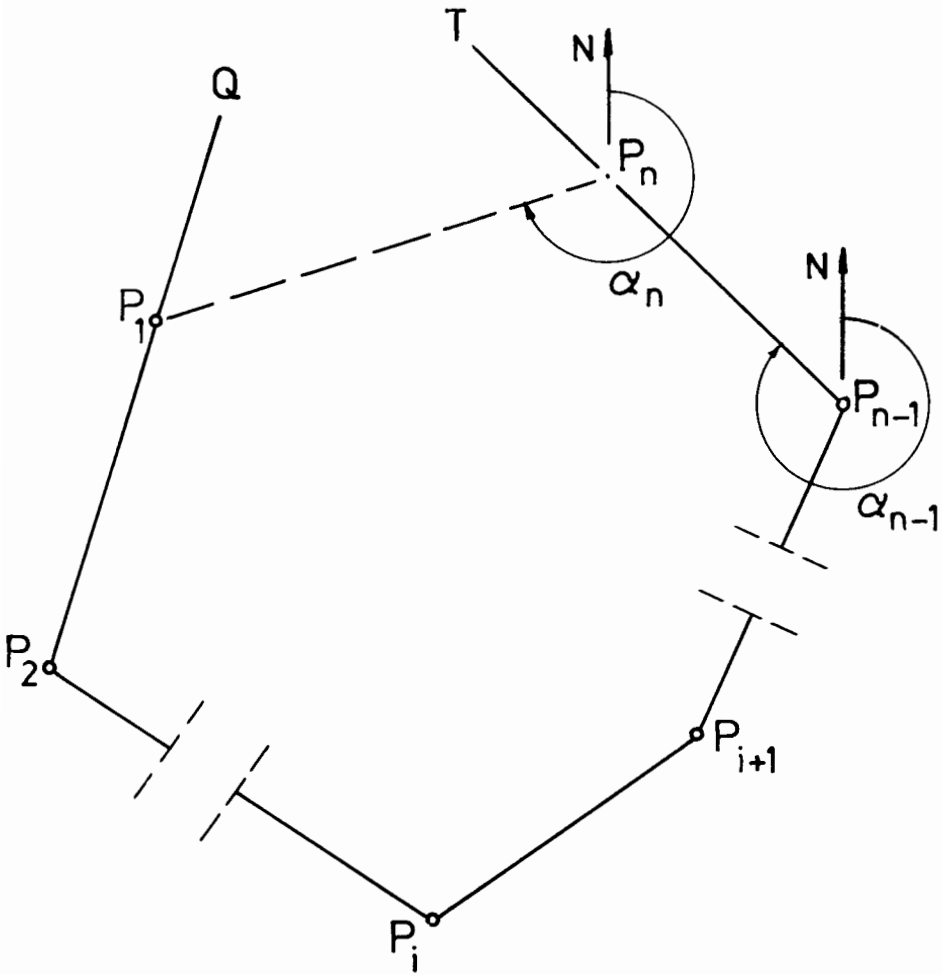


Fig. 2. The coordinates of point P_n are required so that area $P_1P_2 \dots P_iP_{i+1} \dots P_{n-1}P_n$ equals a specified area A .

specified area A by means of a closing line $P_n P_1$. Given the coordinates of point T and points P_1 through point P_{n-1} it is required to calculate the coordinates of point P_n which is the point of intersection of the cutoff line $P_n P_1$ with line $P_{n-1} T$.

The solution requires writing two equations having the unknown coordinates N_n and E_n as parameters. These are the azimuth and area equations.

Azimuth Equation

The relationship between the azimuth α_{n-1} of line $P_{n-1} P_n$ and the coordinates of both points P_{n-1} and P_n is:

$$\tan \alpha_{n-1} = (E_n - E_{n-1}) / (N_n - N_{n-1}) \quad (6)$$

which can also be written in the form:

$$E_n = E_{n-1} + (N_n - N_{n-1}) \tan \alpha_{n-1} \quad (7)$$

Note that $\tan \alpha_{n-1}$ is a known quantity. If it is not given in the data it can be calculated from the coordinates of points P_{n-1} and T as follows:

$$\tan \alpha_{n-1} = (E_T - E_{n-1}) / (N_T - N_{n-1}) \quad (8)$$

Area Equation

The other equation containing both unknown coordinates N_n and E_n is (4) which is called here an area equation. The terms in (4) are divided into two groups; one containing the known and the other the unknown coordinates. Accordingly, (4) takes the following form:

$$2A = 2U + (N_{n-1} + N_n)(E_{n-1} - E_n) + (N_n + N_1)(E_n - E_1) \quad (9)$$

where:

$$2U = \sum_1^{n-2} (N_i + N_{i+1})(E_i - E_{i+1}) \quad (10)$$

The quantity $2U$ can be computed since all coordinates in (10) are given. After multiplying the terms out, (9) reduces to:

$$2A = 2U + E_n(N_1 - N_{n-1}) - N_n(E_1 - E_{n-1}) + N_{n-1}E_{n-1} - N_1E_1 \quad (11)$$

Substituting (7) in (11) gives:

$$2A = 2U + [E_{n-1} + (N_n - N_{n-1}) \tan \alpha_{n-1}](N_1 - N_{n-1}) - N_n(E_1 - E_{n-1}) + N_{n-1}E_{n-1} - N_1E_1 \quad (12)$$

Collecting and arranging terms, then solving for N_n leads to:

$$N_n = \frac{2U - 2A - N_1(E_1 - E_{n-1}) - N_{n-1}(N_1 - N_{n-1}) \tan \alpha_{n-1}}{(E_1 - E_{n-1}) - (N_1 - N_{n-1}) \tan \alpha_{n-1}} \quad (13)$$

from which N_n can be computed. Subsequent substitution in (7) gives E_n .

EXAMPLE NO. 1:

The coordinates of points $P_1, P_2, P_3, P_4,$ and T in Fig. 3 are given in Table 1. Compute the coordinates of points P_5 such that area $P_1P_2P_3P_4P_5$ equals 20585.6 sq. m.

TABLE 1. Data

Point	N m	E m
P_1	+ 22.642	+ 53.927
P_2	- 40.938	+125.156
P_3	+ 2.613	+198.113
P_4	+116.881	+202.153
P_5	?	?
T	+375.521	-225.755

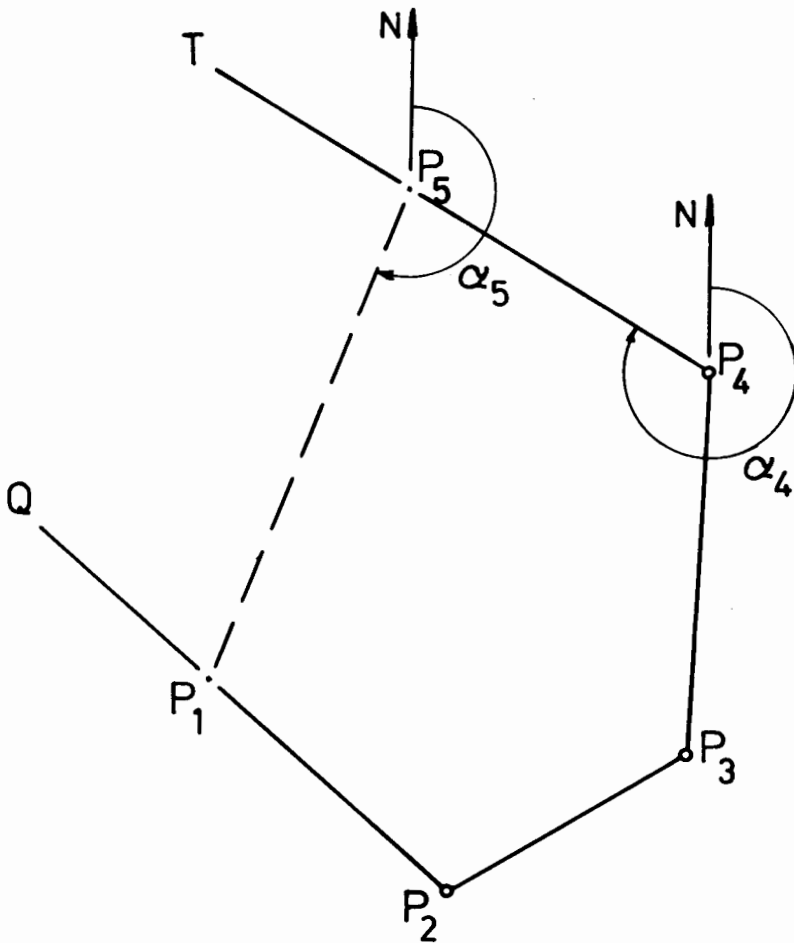


Fig. 3. Cutting off area $P_1P_2P_3P_4P_5$ from tract $QP_2P_3P_4T$.

Solution:

The steps of the solution are:

1. Calculate the partial area $2U$ as shown in Table 2.

TABLE 2. Calculation of partial area $2U$ ($n = 5$)

Point	N m	E m	$N_i + N_{i+1}$ m	$E_i - E_{i+1}$ m	$(N + N)(E - E)$ m^2
P_1	+ 22·642	+ 53·927	— 18·296	— 71·229	+ 1303·2
P_2	— 40·938	+ 125·156	— 38·325	— 72·957	+ 2796·1
P_3	+ 2·613	+ 198·113	+ 119·494	— 4·040	— 482·8
P_4	+ 116·881	+ 202·153			

$$2U = 3616·5 \text{ m}^2$$

2. Calculate $\tan \alpha_{n-1} = \tan \alpha_4$ from the coordinates of points T and P_{n-1} ($P_{n-1} = P_4$).

$$\tan \alpha_{n-1} = \tan \alpha_4 = \frac{E_T - E_4}{N_T - N_4} = \frac{-225·755 - 202·153}{+375·521 - 116·881} = -1·654454$$

3. Calculate N_5 by substituting the data and known quantities, in (13).

$$N_5 = \frac{3616·5 - 2 \times 20585·6 - 22·642 \times (53·927 - 202·153) - 116·881 \times (22·642 - 116·881)(-1·654454)}{(53·927 - 202·153) - (22·642 - 116·881) \times (-1·654454)}$$

$$= 172·361 \text{ m.}$$

4. Substitute in (7) to find E_5 ,

$$E_5 = 202·153 + (172·361 - 116·881) \times (-1·654454) = 110·364 \text{ m.}$$

TABLE 3. Calculation of area $P_1 P_2 P_3 P_4 P_5$

Point	N m	E m	$N_i + N_{i+1}$ m	$E_i - E_{i+1}$ m	$(N + N)(E - E)$ m^2
P_1	+ 22·642	+ 53·927	— 18·296	— 71·229	+ 1303·2
P_2	— 40·938	+ 125·156	— 38·325	— 72·957	+ 2796·1
P_3	+ 2·613	+ 198·113	+ 119·494	— 4·040	— 482·8
P_4	+ 116·881	+ 202·153	+ 289·242	+ 91·789	+ 26549·2
P_5	+ 172·361	+ 110·364	+ 195·003	+ 56·437	+ 11005·4
P_1	+ 22·642	+ 53·927			

$$\text{Double Area} = +41171·1 \text{ m}^2$$

$$\text{Area} = 20585·6 \text{ m}^2$$

$$\text{Desired Area} = 20585·6 \text{ m}^2$$

5. Check the solution by calculating the area $P_1P_2P_3P_4P_5$ using the computed coordinates of point P_5 . The result should be exactly equal to the specified area given. These calculations are shown in Table 3.

CASE 2: CUTOFF LINE HAS A GIVEN BEARING

This problem is demonstrated in Fig. 4. Points $Q, P_1, \dots, P_i, P_{i+1} \dots P_{n-2}$, and T have known coordinates. Line $P_{n-1}P_n$ is known only in direction. It is required to compute the coordinates of points P_{n-1} and P_n so that the area enclosed by the traverse $P_1, \dots, P_i, P_{i+1}, \dots, P_{n-2}P_{n-1}P_n$ equals a specified area A . The four unknown coordinates $E_{n-1}, N_{n-1}, E_n,$ and N_n need four equations for a simultaneous solution. Three azimuth equations and one area equation can be established from the coordinates of the given points as follows:

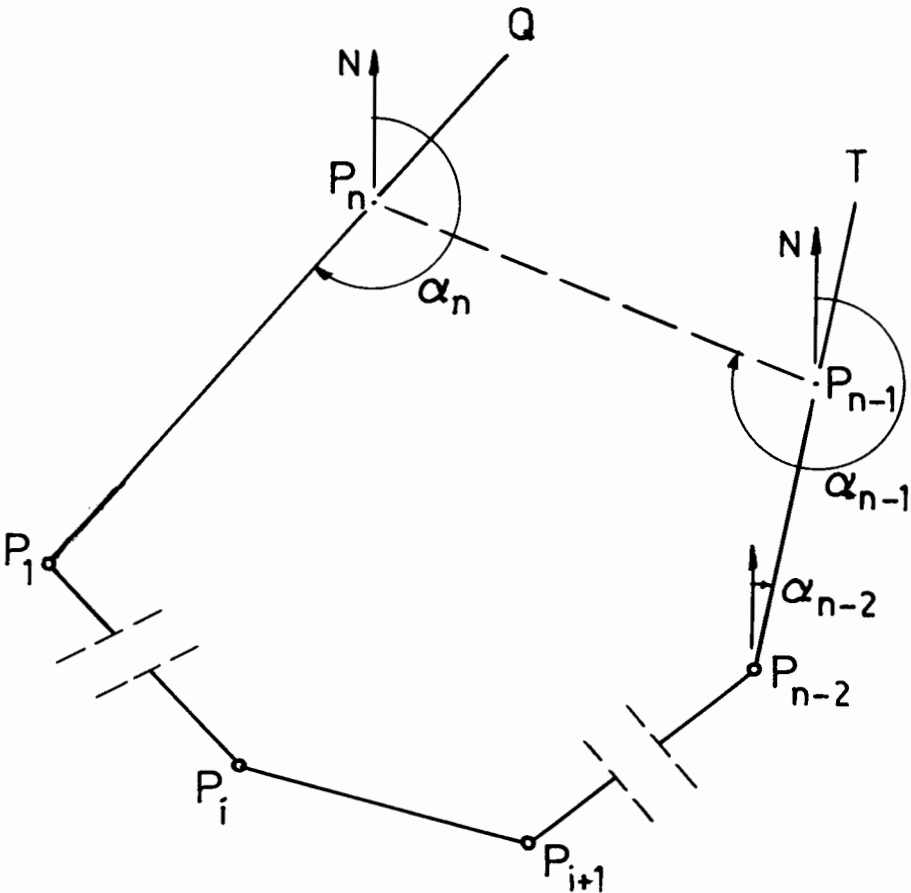


Fig. 4. The coordinates of points P_{n-1} and P_n are required such that line $P_{n-1}P_n$ has a given bearing and area $P_1 \dots P_i P_{i+1} \dots P_{n-2} P_{n-1} P_n$ equals a specified area A .

Azimuth Equations:

$$\tan \alpha_{n-2} = (E_{n-1} - E_{n-2}) / (N_{n-1} - N_{n-2}) \quad (14)$$

$$\tan \alpha_{n-1} = (E_n - E_{n-1}) / (N_n - N_{n-1}) \quad (15)$$

$$\tan \alpha_n = (E_1 - E_n) / (N_1 - N_n) \quad (16)$$

Area Equation

(4) can be written as follows:

$$2A = 2U + (N_{n-2} + N_{n-1})(E_{n-2} - E_{n-1}) + (N_{n-1} + N_n)(E_{n-1} - E_n) \\ + (N_n + N_1)(E_n - E_1) \quad (17)$$

where:

$$2U = \sum_1^{n-3} (N_i + N_{i+1})(E_i - E_{i+1}) \quad (18)$$

All coordinates in (18) are given, therefore $2U$ can be calculated numerically.

The azimuth equations (14), (15), and (16) can be rewritten as follows:

$$E_{n-1} - E_{n-2} = (N_{n-1} - N_{n-2}) \tan \alpha_{n-2} \quad (19)$$

$$E_n - E_{n-1} = (N_n - N_{n-1}) \tan \alpha_{n-1} \quad (20)$$

$$E_1 - E_n = (N_1 - N_n) \tan \alpha_n \quad (21)$$

Adding equations (19), (20), and (21) will eliminate the unknown coordinates E_{n-1} and E_n . The summation gives after arranging terms:

$$E_1 - E_{n-2} = (\tan \alpha_{n-2} - \tan \alpha_{n-1}) \times N_{n-1} + (\tan \alpha_{n-1} - \tan \alpha_n) \times N_n \\ + N_1 \tan \alpha_n - N_{n-2} \tan \alpha_{n-2} \quad (22)$$

where N_{n-1} and N_n are the only unknown quantities. (22) leads to:

$$N_n = \frac{b' - (\tan \alpha_{n-2} - \tan \alpha_{n-1}) \times N_{n-1}}{\tan \alpha_{n-1} - \tan \alpha_n} \quad (23)$$

where:

$$b' = E_1 - N_1 \tan \alpha_n - E_{n-2} + N_{n-2} \tan \alpha_{n-2} \quad (24)$$

Also, one obtains from (21):

$$E_n = E_1 - N_1 \tan \alpha_n + N_n \tan \alpha_n \quad (25)$$

Substituting (23) in (25) gives:

$$E_n = E_1 - N_1 \tan \alpha_n + \frac{\tan \alpha_n}{\tan \alpha_{n-1} - \tan \alpha_n} [b' - (\tan \alpha_{n-2} - \tan \alpha_{n-1})N_{n-1}] \quad (26)$$

(19) can be rewritten as follows:

$$E_{n-1} = E_{n-2} + N_{n-1} \tan \alpha_{n-2} - N_{n-2} \tan \alpha_{n-2} \quad (27)$$

(23), (26), and (27) give the unknown coordinates N_n , E_n , and E_{n-1} as functions of N_{n-1} .

After multiplication (17) reduces to:

$$\begin{aligned} 2A = 2U + N_{n-2}E_{n-2} + N_{n-1}E_{n-2} - N_{n-2}E_{n-1} + N_nE_{n-1} \\ - N_{n-1}E_n + N_1E_n - N_nE_1 - N_1E_1 \end{aligned} \quad (28)$$

Substituting for N_n , E_{n-1} , and E_n by their values as found in (23), (26), and (27) will give one equation with only one unknown coordinate, namely, N_{n-1} . After substitution, the unknown terms of (28) become:

$$N_{n-2}E_{n-1} = N_{n-2}E_{n-2} - N_{n-2}^2 \tan \alpha_{n-2} - N_{n-2} \tan \alpha_{n-2} N_{n-1} \quad (29)$$

$$\begin{aligned} N_nE_{n-1} = [b'(E_{n-2} - N_{n-2} \tan \alpha_{n-2}) + \{b' \tan \alpha_{n-2} - (\tan \alpha_{n-2} - \tan \alpha_{n-1}) \\ \times (E_{n-2} - N_{n-2} \tan \alpha_{n-2})\} \times N_{n-1}] / (\tan \alpha_{n-1} - \tan \alpha_n) \\ - (\tan \alpha_{n-2} - \tan \alpha_{n-1}) \times \tan \alpha_{n-2} N_{n-1}^2 / (\tan \alpha_{n-1} - \tan \alpha_n) \end{aligned} \quad (30)$$

$$\begin{aligned} N_{n-1}E_n = [E_1 - N_1 \tan \alpha_n + b' \tan \alpha_n / (\tan \alpha_{n-1} - \tan \alpha_n)] N_{n-1} \\ - (\tan \alpha_{n-2} - \tan \alpha_{n-1}) \tan \alpha_n N_{n-1}^2 / (\tan \alpha_{n-1} - \tan \alpha_n) \end{aligned} \quad (31)$$

$$\begin{aligned} N_1E_n = N_1E_1 - N_1^2 \tan \alpha_n + b'N_1 \tan \alpha_n / (\tan \alpha_{n-1} - \tan \alpha_n) \\ - (\tan \alpha_{n-2} - \tan \alpha_{n-1})N_1 \tan \alpha_n N_{n-1} / (\tan \alpha_{n-1} - \tan \alpha_n) \end{aligned} \quad (32)$$

$$\begin{aligned} N_nE_1 = b'E_1 / (\tan \alpha_{n-1} - \tan \alpha_n) \\ - (\tan \alpha_{n-2} - \tan \alpha_{n-1})E_1 N_{n-1} / (\tan \alpha_{n-1} - \tan \alpha_n) \end{aligned} \quad (33)$$

Substituting (29) through (33) in (28), collecting and arranging terms will lead to the following equation of the second degree:

$$aN_{n-1}^2 + bN_{n-1} + c = 0 \quad (34)$$

in which the coefficients a , b , and c are:

$$a = \frac{(\tan \alpha_{n-2} - \tan \alpha_{n-1})(\tan \alpha_n - \tan \alpha_{n-2})}{\tan \alpha_{n-1} - \tan \alpha_n} \tag{35}$$

$$b = 2b' \frac{\tan \alpha_{n-2} - \tan \alpha_{n-1}}{\tan \alpha_{n-1} - \tan \alpha_n} \tag{36}$$

$$c = - \frac{b'}{\tan \alpha_{n-1} - \tan \alpha_n} - N_1^2 \tan \alpha_n + N_{n-2}^2 \tan \alpha_{n-2} + 2U - 2A \tag{37}$$

and b' is as given in (24).

(34) gives two solutions for N_{n-1} , one of which fits the given data. The other required coordinates N_n , E_n , and E_{n-1} can be obtained by substituting the proper value of N_{n-1} in (23), (26), and (27) respectively. The calculations are then checked by computing the area $P_1 P_2 \dots P_{n-2} P_{n-1} P_n$ using the already found coordinates of points P_{n-1} and P_n . The result is then compared with the desired area. Both must of course be equal.

EXAMPLE NO. 2:

It is required to subdivide area $ABCDE$ into two equal areas by line PQ parallel to side CD as shown in Fig. 5. Given the coordinates of points A , B , C , D , and E (Table 4), compute the coordinates of points P and Q .

TABLE 4. Data

Point	N m	E m
A	411.95	103.41
B	929.10	564.81
C	978.15	739.56
D	762.08	816.37
E	220.65	315.74

Solution:

Area calculation by coordinates shows that the area of tract $ABCDE$ is 229707.7 m^2 . The desired area $EAPQ$ is therefore $229707.7/2 = 114853.8 \text{ m}^2$. Using (4) and noting that the order of the points E , A , P , and Q is in a clockwise direction; i.e. area is negative, then:

$$2A = -2 \times \frac{229707.7}{2} = -229707.7 \text{ m}^2$$

The steps of the solution are the calculations of:

Partial Area 2U: The partial area $2U$ is calculated from the coordinates of the given points E and A by (18). The calculation is shown in Table 5.

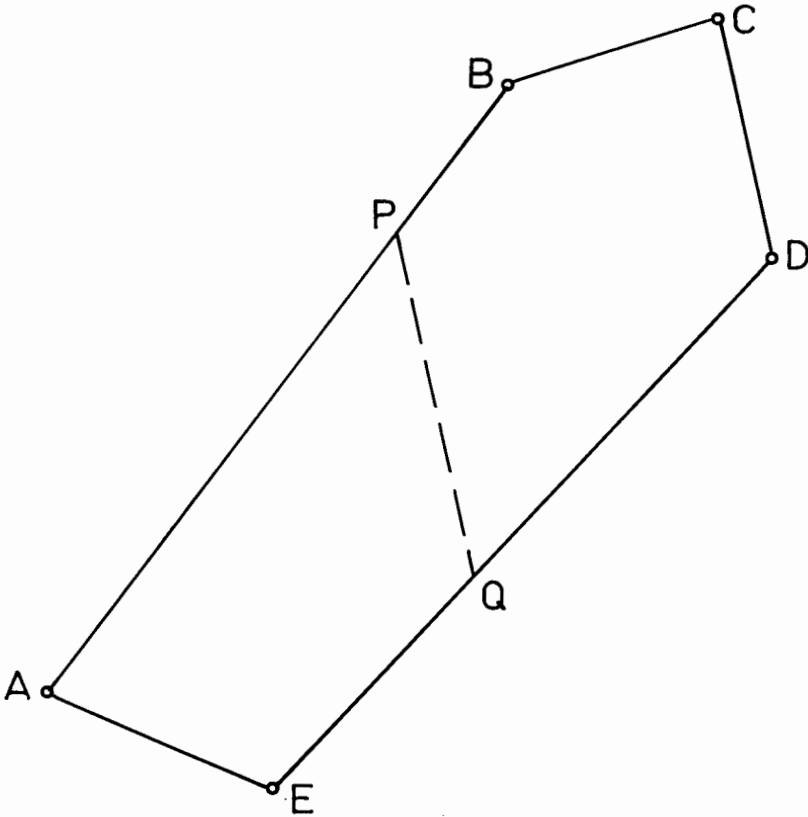


Fig. 5. PQ is parallel to CD and divides the area $ABCDE$ into two equal parts.

TABLE 5. Calculation of partial area $2U$

Point	N m	E m	$N_E + N_A$ m	$E_E - E_A$ m	$(N + N)(E - E)$ m^2
E	220.65	315.74			
A	411.95	103.41	+632.60	+212.33	+134320.0
					$2U = +134320.0$

Tangents of the Azimuths: Considering that points $E, A, P,$ and Q stand for $P_1, P_{n-2}, P_{n-1},$ and $P_n,$ then:

$$\tan \alpha_{n-2} = \tan (AB) = \frac{564.81 - 103.41}{929.10 - 411.95} = +0.892198$$

$$\tan \alpha_{n-1} = \tan (CD) = \frac{816.37 - 739.56}{762.08 - 978.15} = +0.355487$$

$$\tan \alpha_n = \tan (DE) = \frac{315.74 - 816.37}{220.65 - 762.08} = +0.924644$$

Differences in the Tangents of the Azimuths:

$$\tan(AB) - \tan(CD) = +0.892198 + 0.355487 = +1.247685$$

$$\tan(CD) - \tan(DE) = -0.355487 - 0.924644 = -1.280131$$

$$\tan(DE) - \tan(AB) = +0.924644 - 0.892198 = +0.032446$$

$$\Sigma = 0.0 \quad (\text{Check})$$

Coefficients of the Second Degree Equation: The coefficients a , b' , b , and c are obtained from (35), (24), (36), and (37), respectively.

$$a = \frac{1.247685 \times 0.032446}{-1.280131} = -0.031624$$

$$b' = 315.74 - 220.65 \times 0.924644 - 103.41 + 411.95 \times 0.892198 = 375.848 \text{ m}$$

$$b = 2 \times 375.848 \times \frac{1.247685}{-1.280131} = -732.644 \text{ m}$$

$$c = -\frac{375.848^2}{-1.280131} - 220.65^2 \times 0.924644 + 411.95^2 \times 0.892198 \\ + 134320.0 - (-229707.7) = +580768.0 \text{ m}^2$$

Required Coordinates:

Substituting in (34) gives

$$-0.031624 N_{n-1}^2 - 732.644 N_{n-1} + 580768.0 = 0$$

from which N_{n-1} is found to be:

$$N_p = N_{n-1} = 767.289 \text{ m}$$

Substituting in (23), (26), and (27) gives:

$$N_Q = N_n = \frac{375.848 - 1.247685 \times 767.289}{-1.280131} = 454.241 \text{ m}$$

$$E_Q = E_n = 315.74 - (220.65 - 454.24) \times 0.924644 = 531.728 \text{ m}$$

$$E_p = E_{n-1} = 103.41 + (767.29 - 411.95) \times 0.892198 = 420.444 \text{ m}$$

Check:

The solution is checked by first calculating the azimuth of line PQ and comparing it with the given azimuth of line CD .

$$\tan(PQ) = \frac{531.728 - 420.444}{454.241 - 767.289} = -0.355485$$

$$\tan(CD) = -0.355487 \text{ (Check)}$$

The second check is to calculate the area $EAPQ$ using the coordinates of points P and Q as found. This calculation is shown in Table 6.

The small difference of 0.7 m^2 between the calculated and the desired areas resulted from rounding the computed coordinates of points P and Q to two decimals. This difference is apparently insignificant.

TABLE 6. Area of parcel $EAPQ$

Point	N m	E m	$N_i + N_{i+1}$ m	$E_i + E_{i+1}$ m	$(N+N)(E-E)$ m^2
E	220.65	315.74			
A	411.95	103.41	+ 632.60	+ 212.33	+ 134320.0
P	767.29	420.44	+ 1179.24	- 317.03	- 373854.5
Q	454.24	531.73	+ 1221.53	- 111.29	- 135944.1
E	110.65	315.74	+ 674.89	+ 215.99	+ 145769.5
					Double Area = -229709.1 m^2
					Area = 114854.5 m^2
					Desired Area = 114853.8 m^2
					Difference = 0.7 m^2

SUMMARY

The solutions of the two cases presented here are based on writing an area equation where the coordinates of one or both endpoints of the cutoff line are unknown. Sufficient azimuth equations having the same unknown coordinates as parameters are readily available. Solving these equations simultaneously leads to formulae for the desired coordinates as functions of the given data. Two numerical examples are given to demonstrate the solution.

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Reference

1. Danial, Naguib F., 1982. Land Partitioning by Double Meridian Distance Method. Technical Papers of the 1982 ACSM-ASP Fall Convention, pp.91-100.