# Another Solution of the Three-Point Problem

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ABSTRACT. In the three-point problem, also called the resection problem, three directions are measured from a new point towards three points, the positions of which are known. The figure formed by the measured and the known directions is a quadrilateral. Two of its angles are observed, two can be found from the known position of the points, and the remaining four angles are unknown. The principle of the method described here is to solve the four condition equations of the quadrilateral simultaneously thereby determining the values of the unknown angles.

## General

The need frequently arises to determine the location of an unknown point $P$ by occupying the point and measuring the horizontal angles $\alpha$ and $\beta$ between three visible and known stations such as $A$, $B$, and $C$ as shown in Figure 1. Direct computation of the position of point $P$ from the given data and the observations is not possible and therefore lengthy calculations are required. The aim of these calculations is to determine the angles $\beta_1$ and $\alpha_3$ (Fig. 1) so that point $P$ can be located by intersection.

There are various analytical solutions to this problem (Allan, *et al.*, p. 293, and Jordan, *et al.*, p. 40).

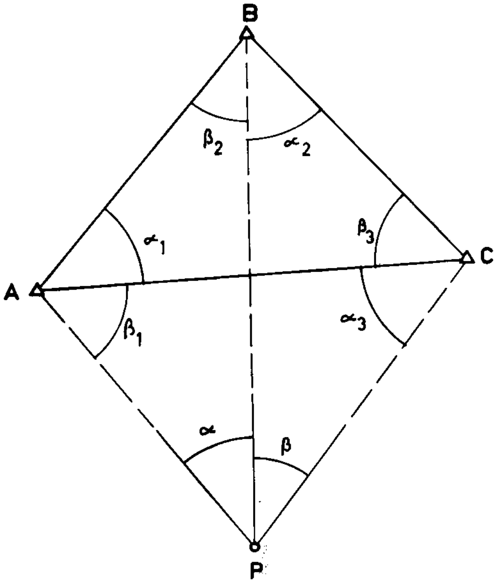


FIGURE 1. The new point $P$ is outside the triangle $ABC$ and on the opposite side of the middle point $B$.

## Mathematical proof of a new method

In Figure 1, the given points and the new point are lettered $A, B, C$, and $P$ respectively in a clockwise direction. The angles between the different lines are given the symbols shown in the figure.

The triangle $ABC$ is known either by the distances $AB$, $BC$, and the included angle $C\hat{B}A$ or more usually by the coordinates of its points $A, B$, and $C$. In either case the internal angles $\alpha_1$, $(\alpha_2 + \beta_2)$, and $\beta_3$ can be readily found. Noting that $\alpha$ and $\beta$ are observed at point $P$, the remaining angles $\beta_1$, $\alpha_2$, $\beta_2$, and $\alpha_3$ will be the only unknown angles of the quadrilateral.

The angles in a quadrilateral are related to each other by four conditions which result from three independent angle equations and one side equation.

*The angle equations*

For triangle $PAC$

$$\alpha + \beta + \beta_1 + \alpha_3 - 180° = 0 \quad (1)$$

For triangle $PAB$

$$\alpha + \alpha_1 + \beta_1 + \beta_2 - 180° = 0 \quad (2)$$

For triangle $ABC$

$$\alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 180° = 0 \quad (3)$$

*The side equation*

Assume one line, as $AB$, is known and calculate another value for it by solving a series of triangles. The calculations are carried out as follows:

$$AB = AC \frac{\sin \beta_3}{\sin(\alpha_2 + \beta_2)} \quad \text{from triangle } ABC$$

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$$AC = AP \frac{\sin(\alpha + \beta)}{\sin \alpha_3} \quad \text{from triangle } APC$$

$$AP = AB \frac{\sin \beta_2}{\sin \alpha} \quad \text{from triangle } APB$$

It is readily seen from the above mentioned equations that

$$AB = AB \frac{\sin \beta_3}{\sin(\alpha_2 + \beta_2)} \cdot \frac{\sin(\alpha + \beta)}{\sin \alpha_3} \cdot \frac{\sin \beta_2}{\sin \alpha} \tag{4a}$$

which can be written after rearranging the terms as follows:

$$\frac{\sin(\alpha + \beta)}{\sin(\alpha_2 + \beta_2)} \cdot \frac{\sin \beta_3}{\sin \alpha} \cdot \frac{\sin \beta_2}{\sin \alpha_3} = 1 \tag{4b}$$

The same result can be obtained in a quicker way by choosing point $A$ as a pole. The sides $AB$, $AC$, and $AP$, which are radiating from this pole, are written in order. By writing them again underneath in the same order but starting from the second side $AC$ and equating the resulting expression to unity, equation (4c) results.

$$\frac{AB}{AC} \cdot \frac{AC}{AP} \cdot \frac{AP}{AB} = 1 \tag{4c}$$

Every quotient in equation (4c) contains two sides of a triangle. Replacing these sides by the sines of the angles opposite them in these triangles and rearranging terms will lead to the same equation (4b).

The only unknowns in equation (4b) are $\beta_2$ and $\alpha_3$. It can therefore be reduced to the following form.

$$m \cdot n_a \cdot \frac{\sin \beta_2}{\sin \alpha_3} = 1$$

or

$$K_a \cdot \frac{\sin \beta_2}{\sin \alpha_3} = 1 \tag{5}$$

where

$$m = \frac{\sin(\alpha + \beta)}{\sin(\alpha_2 + \beta_2)} \tag{6}$$

$$n_a = \frac{\sin \beta_3}{\sin \alpha} \tag{7}$$

and

$$K_a = m \cdot n_a \tag{8}$$

From equations (1) and (2) it is seen that

$$\alpha_3 = (180° - \alpha - \beta) - \beta_1 = \theta - \beta_1 \tag{9}$$

and

$$\beta_2 = (180° - \alpha - \alpha_1) - \beta_1 = \phi - \beta_1 \tag{10}$$

where

$$\theta = 180° - \alpha - \beta \tag{11}$$

$$\phi = 180° - \alpha - \alpha_1 \tag{12}$$

Substituting these values into equation (5), the side equation then becomes

$$K_a \cdot \frac{\sin(\phi - \beta_1)}{\sin(\theta - \beta_1)} = 1 \tag{13a}$$

or

$$K_a (\sin\phi \cos\beta_1 - \cos\phi \sin\beta_1) = \sin\theta \cos\beta_1 - \cos\theta \sin\beta_1$$

After collecting like terms

$$\cos\beta_1 (K_a \cdot \sin\phi - \sin\theta) = \sin\beta_1 (K_a \cdot \cos\phi - \cos\theta)$$

and this gives

$$\tan\beta_1 = \frac{K_a \cdot \sin\phi - \sin\theta}{K_a \cdot \cos\phi - \cos\theta} \tag{13b}$$

from which the angle $\beta_1$ can be calculated. By substituting back into equations (9), (2), and (3) the other angles $\alpha_3$, $\beta_2$, and $\alpha_2$ can be determined. It is sufficient, however, to calculate only the angle $\alpha_3$ because the angles $\alpha_2$ and $\beta_2$ are not required for further calculations.

The computations can be checked by calculating the angle $\alpha_3$ independently. This is possible if another side equation with pole at $C$ is considered. Following the procedure described previously the following equation will be obtained.

$$\frac{CB}{CA} \cdot \frac{CA}{CP} \cdot \frac{CP}{CB} = 1$$

which leads to

$$\frac{\sin(\alpha + \beta)}{\sin(\alpha_2 + \beta_2)} \cdot \frac{\sin \alpha_1}{\sin \beta} \cdot \frac{\sin \alpha_2}{\sin \beta_1} = 1 \tag{14}$$

or

$$m \cdot n_c \cdot \frac{\sin \alpha_2}{\sin \beta_1} = 1$$

$$K_c \cdot \frac{\sin \alpha_2}{\sin \beta_1} = 1 \tag{15}$$

where

$m$ = the same factor mentioned in equation (6)

$$n_c = \frac{\sin \alpha_1}{\sin \beta} \tag{16}$$

$$K_c = m \cdot n_c \tag{17}$$

The angle equation obtained from the triangle $PBC$ (Fig. 1) is

$$\beta + \alpha_2 + \alpha_3 + \beta_3 - 180° = 0$$

from which

$$\alpha_2 = (180° - \beta - \beta_3) - \alpha_3 = \psi - \alpha_3 \qquad (18)$$

Similarly from (1)

$$\beta_1 = (180° - \alpha - \beta) - \alpha_3 = \theta - \alpha_3 \qquad (19)$$

where

$$\psi = 180° - \beta - \beta_3 \qquad (20)$$

and

$$\theta = 180° - \alpha - \beta$$

as is described by eq. (11)

The following equation can be derived in a similar manner to equation (13b).

$$\tan \alpha_3 = \frac{K_c \cdot \sin\psi - \sin\theta}{K_c \cdot \cos\psi - \cos\theta} \qquad (21)$$

From which angle $\alpha_3$ can be computed.

As a check, the angles in the triangle *PAC* should add up to 180°, or in other words, equation (1) must be satisfied.

The equations derived are valid for all cases of resection. If the new point *P* lies within the triangle *ABC*, as shown in Figure 2, the values obtained for the angles $\beta_1$ and $\alpha_3$ will be negative. However, for the case when the new point lies outside the triangle but on the same side as the middle point *B*, (see Figure 3), the known angles $\alpha_1$ and $\beta_3$ should be considered negative. Accordingly equations 12 and 20 will take the following form

$$\phi = 180° - \alpha + \alpha_1 \qquad (12')$$

$$\psi = 180° - \beta + \beta_3 \qquad (20')$$

## Summary of the derived formulae

Considering the notation shown on Figures 1, 2, and 3, the problem and the solution can be summarized in the following:

given: $\alpha_1$, $(\alpha_2 + \beta_2)$, and $\beta_3$

measured: $\alpha$ and $\beta$

required: $\beta_1$ and $\alpha_3$

solution:

$$m = \frac{\sin(\alpha + \beta)}{\sin(\alpha_2 + \beta_2)} \qquad (6)$$

$$n_a = \frac{\sin\beta_3}{\sin\alpha} \qquad (7)$$

$$K_a = m \cdot n_a \qquad (8)$$

$$\theta = 180° - \alpha - \beta \qquad (11)$$

$$\phi = 180° - \alpha \pm \alpha_1 \qquad (12)$$

$$\beta_1 = \tan^{-1}\frac{K_a \cdot \sin\phi - \sin\theta}{K_a \cdot \cos\phi - \cos\theta} \qquad (13a)$$

$$\alpha_3 = 180° - \alpha - \beta - \beta_1 \qquad (1)$$

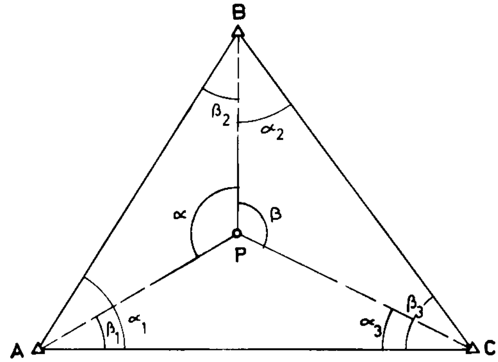


FIGURE 2. The new point *P* is inside the triangle *ABC*.

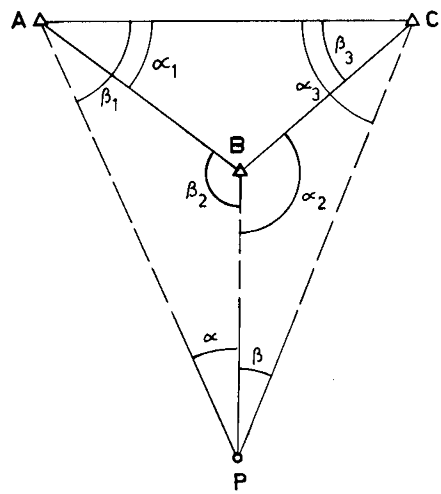


FIGURE 3. The new point *P* is outside the triangle *ABC* but on the same side of the middle point *B*.

Checking:

$$n_c = \frac{\sin\alpha_1}{\sin\beta} \qquad (16)$$

$$K_c = m \cdot n_c \qquad (17)$$

$$\psi = 180° - \beta \pm \beta_3 \qquad (20)$$

$$\alpha_3 = \tan^{-1}\frac{K_c \cdot \sin\psi - \sin\theta}{K_c \cdot \cos\psi - \cos\theta} \qquad (21)$$

*Notes:*

1. The positive and negative signs in equations (12) and (20) correspond to the cases when $(\alpha_2 + \beta_2)$ is greater or smaller than 180° respectively.

2. $\alpha_1$ and $\beta_3$ will be negative in the case when the new point lies inside the triangle *ABC*.

## The coordinates of the new point P

It is generally required to calculate the coordinates of the new point *P* when the coordinates of points $A$, $B$, and $C$ are given. In this case the distances $AP$ and $CP$ can be obtained from solving the triangle $ACP$. The following equations do not need any further explanation.

$$\tan(AC) = \frac{E_C - E_A}{N_C - N_A} \tag{22}$$

where $(AC)$ is the azimuth of the line from $A$ to $C$.

$$(AP) = (AC) + \beta_1 \tag{23}$$

$$(CP) = (AC) - \alpha_3 \pm 180° \tag{23'}$$

$$\left.\begin{aligned} N_{P_A} &= N_A + AP\cos(AP) \\ E_{P_A} &= E_A + AP\sin(AP) \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} N_{P_C} &= N_C + CP\cos(CP) \\ E_{P_C} &= E_C + CP\sin(CP) \end{aligned}\right\} \tag{24'}$$

**Numerical example**

Table 1 shows a computer output of a problem and its detailed solution. It is sufficient, however, to carry out the calculations with six decimal figures on a hand-held scientific calculator as follows:

Measurements:

$$\begin{aligned} \alpha &= 111°27'32'' = 111.458\,889° \\ \beta &= 124\ 38\ 16 = 124.637\,778 \\ \alpha + \beta &= 236.096\,667 \end{aligned}$$

Obtainable angles from the given coordinates of $A$, $B$, and $C$:

$$\begin{aligned} \alpha_1 &= 52.338\,580° \\ (\alpha_2 + \beta_2) &= 49.934\,466 \\ \beta_3 &= 77.726\,954 \end{aligned}$$

Calculations:

$$m = \frac{\sin(\alpha + \beta)}{\sin(\alpha_2 + \beta_2)} = -1.084\,503$$

$$n_a = \frac{\sin\beta_3}{\sin\alpha} = 1.049\,926$$

$$K_a = m \cdot n_a = -\ 1.138\,648$$

$$\theta = 180° - \alpha - \beta = -56.096\,667°$$

$$\phi = 180° - \alpha - \alpha_1 = 16.202\,531°$$

$$\tan\beta_1 = \frac{K_a \cdot \sin\phi - \sin\theta}{K_a \cdot \cos\phi - \cos\theta} = \frac{0.512\,259}{-1.651\,217} = -0.310\,231$$

$$\beta_1 = -17.235\,510° = -17°14'08''$$

## Table No. 1

DATA

| Pt. | N | E | Angle | | |
|---|---|---|---|---|---|
| | m | m | | | |
| A | -58956.700 | 86600.740 | 111°27'32'' | | |
| B | -53275.440 | 88961.820 | 124 38 16 | | |
| C | -57701.920 | 91253.070 | | | |

| | ANGLE | | SIN | COS | CONSTANTS |
|---|---|---|---|---|---|
| $\alpha+\beta$ | 236° 5'48'' | 236.0966667° | -0.8299801 | | |
| $\alpha_2+\beta_2$ | 49 56 4 | 49.9344662 | 0.7653088 | | m = -1.0845035 |
| $\beta_3$ | 77 43 37 | 77.7269535 | 0.9771457 | | $n_a$ = 1.0499264 |
| $\alpha$ | 111 27 32 | 111.4588889 | 0.9306802 | | $K_a$ = -1.1386488 |
| $\phi$ | 16 12 9 | 16.2025309 | 0.2790336 | 0.9602814 | |
| | | | -0.3177212 | -1.0934232 | |
| $\theta$ | -56 -5-48 | -56.0966667 | -0.8299799 | 0.5577933 | |
| $\beta_1$ | | | 0.5122587 | -1.6512166 | -0.3102311 ($\tan\beta_1$) |
| $\beta_1$ | -17-14 -8 | -17.2355117 | | | |
| $\alpha_1$ | 52 20 19 | 52.3385802 | 0.7916352 | | $n_c$ = 0.9621683 |
| $\beta$ | 124 38 16 | 124.6377778 | 0.8227617 | | $K_c$ = -1.0434749 |
| $\psi$ | -22-21-53 | -22.3647313 | -0.3805012 | 0.9247804 | |
| | | | 0.3970435 | -0.9649852 | |
| $\theta$ | -56 -5-48 | -56.0966667 | -0.8299799 | 0.5577933 | |
| $\alpha_3$ | | | 1.2270234 | -1.5227785 | -0.8057793 ($\tan\alpha_3$) |
| $\alpha_3$ | -38-51-40 | -38.8611444 | | | |

Check

ALFA = 111° 27'32''
BETA = 124 38 16
BETA 1 = -17-14 -8
ALFA 3 = -38-51-40

SUM = 180 00 00

**The coordinates of the new point P**

NORTHING = -57008.641 m : EASTING = 89678.743 m

$\alpha_3$ can be obtained directly by substituting $\beta_1$ in equation (9), or can be calculated in a similar way as shown above. In the second case the computations are checked when equation (1) is satisfied.

Further calculations will show that

$$\begin{array}{rlrl} (AC) = & 74.905\,912^\circ; & (CA) = & 254.905\,912^\circ \\ +\beta_1 = & -17.235\,510^\circ; & -\alpha_3 = & +38.861\,157^\circ \\ \hline (AP) = & 57.670\,402^\circ & (CP) = & 293.767\,069^\circ \end{array}$$

and

$$AC = 4818.573\text{ m}, AP = 3642.670\text{ m}, \text{ and } CP = 1720.215\text{ m}.$$

By substitution in either set of equations (24) or (24′) the coordinates of the new point $P$ will be determined. They are exactly the same as shown in Table 1.

## Conclusion

In this paper a new method is presented of solving the three-point problem. The mathematical proof and a numerical example are given. The method is suitable for calculations using hand-held scientific calculators.

## REFERENCES

1. Allan, A. L., J. R. Hollwey, and J. H. B. Maynes, *Practical Field Surveying and Computations,* Heineman, London, 1973.

2. Jordan, Eggert und Kneissl, *Handbuch der Vermessungskunde,* Vol. 2, J. B. Metzlersche Verlagsbuchhandlung, Stuttgart, 1963. ■