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THREE POINT RESECTION PROBLEM

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INTRODUCTION

The three-point resection problem in surveying involves occupying an unknown point and observing angles only to three known points. Today, with the advent of total stations/EDMs, the problem is greatly simplified. If the unknown point P lies on a circle defined by the three known control points then the solution is indeterminate or not uniquely possible. There are, theoretically, an infinite number of solutions for the observed angles. If the geometry is close to this, then the solution is weak. In addition, there is no solution to this problem when all the points lie on a straight or nearly straight line. There are a number of approaches to solving the resection problem.

KAESTNER-BURKHARDT METHOD

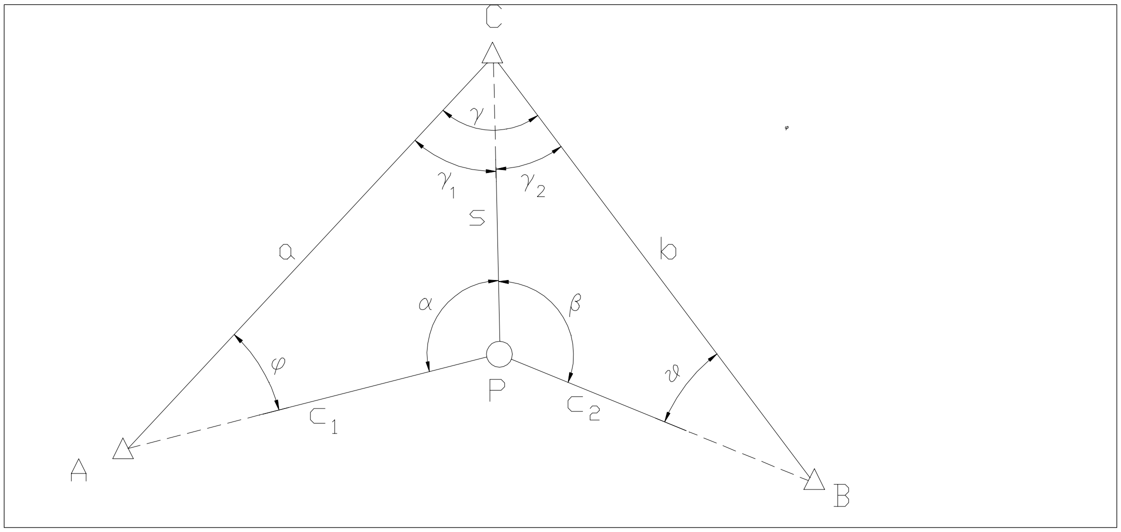


Figure 1. Three point resection problem using the Kaestner-Burkhardt method.

In the Kaestner-Burkhardt approach [Blachut et al, 1979, Faig, 1972, Kissam, 1981, Ziemann, 1974] (also referred to as the Pothonot-Snellius method [Allan et. al., 1968]) the coordinates of points A, B, and C are known and the angles α and β measured at point P. Inversing between the control points we can compute a, b, Az_{AC}, and Az_{BC} using the following relationships:

$$\mathrm{Az}_{AC} = \tan^{-1}\left(\frac{X_C - X_A}{Y_C - Y_A}\right) \qquad a = \sqrt{(X_C - X_A)^2 + (Y_C - Y_A)^2}$$

$$\mathrm{Az}_{BC} = \tan^{-1}\left(\frac{X_C - X_B}{Y_C - Y_B}\right) \qquad b = \sqrt{(X_C - X_B)^2 + (Y_C - Y_B)^2}$$

Compute γ

$$\gamma = \mathrm{Az}_{CA} - \mathrm{Az}_{CB}$$

$$= \mathrm{Az}_{AC} - \mathrm{Az}_{BC}$$

Compute the auxiliary angles φ and θ. First, recognize that the sum of the interior angles is equal to 360^o [the sum of interior angles of a polygon must equal $(n – 2)180^o$].

$$\phi + \alpha + \beta + \theta + \gamma = 360^o$$

Rearrange

$$\frac{1}{2}(\phi + \theta) = 180^o - \frac{1}{2}(\alpha + \beta + \gamma) = \delta_1$$

From the sine rule, compute the distance s

$$s = \frac{a \sin \phi}{\sin \alpha} \qquad \text{and} \qquad s = \frac{b \sin \theta}{\sin \beta}$$

Combining these relationships yields

$$\frac{\sin \phi}{\sin \theta} = \frac{b}{a}\frac{\sin \alpha}{\sin \beta} = \cot \lambda$$

where λ is an auxiliary angle with an uncertainty of $\pm 180^o$. We then have

$$\frac{\sin \phi}{\sin \theta} = \cot \lambda$$

or

$$\frac{\sin \phi - \sin \theta}{\sin \phi + \sin \theta} = \frac{\cot \lambda - 1}{\cot \lambda + 1}$$

Since $\cot\lambda = \frac{\cot\lambda}{1}$ and using trigonometric theorems, one can write

$$\frac{2\cos\frac{1}{2}(\phi+\theta)\sin\frac{1}{2}(\phi-\theta)}{2\sin\frac{1}{2}(\phi+\theta)\cos\frac{1}{2}(\phi-\theta)} = \frac{\cot 45^o \cot\lambda - 1}{\cot\lambda + \cot 45^o}$$

But, recognizing that cot 45^o = 1 and

$$\tan\frac{1}{2}(\phi-\theta) = \tan\frac{1}{2}(\phi-\theta)\cot\left(45^o+\lambda\right) = \tan\delta_1 \cot\left(45^o+\lambda\right)$$

Therefore,

$$\frac{1}{2}(\phi-\theta) = \tan^{-1}\left[\tan\delta_1\cot\left(45^o+\lambda\right)\right] = \delta_2$$

Then,

$$\phi = \delta_1 + \delta_2$$

$$\theta = \delta_1 - \delta_2$$

Recall that δ_2 has an uncertainty of ±180^o due to the uncertainty in λ. Next, using the sine rule, compute the distances c_1 and c_2.

$$c_1 = a\frac{\sin\gamma_1}{\sin\alpha} = a\frac{\sin\left[180^o - (\alpha+\phi)\right]}{\sin\alpha} = a\frac{\sin(\alpha+\phi)}{\sin\alpha}$$

$$c_2 = b\frac{\sin\gamma_2}{\sin\beta} = b\frac{\sin\left[180^o - (\beta+\theta)\right]}{\sin\beta} = b\frac{\sin(\beta+\theta)}{\sin\beta}$$

If λ was picked in the right quadrant then γ_2 is in the right quadrant and c_1 and c_2 are positive. If they turn out to be negative, δ_2, ϕ, and θ have to be changed by 180^o. As a check, recall that $\alpha + \beta + \gamma + \phi + \theta = 360°$. The next step is to compute the azimuths to point P.

$$Az_{AP} = Az_{AC} + \phi$$

$$Az_{BP} = Az_{BC} - \theta$$

Finally, compute the coordinates of point P.

$$X_P = X_A + c \sin Az_{AP} = X_B + c_2 \sin Az_{BP}$$

$$Y_P = Y_A + c_1 \cos Az_{AP} = Y_B + c_2 \cos Az_{BP}$$

An example, prepared using Mathcad is presented as follows.

Three Point Resection Problem
Kaestner-Burkhardt Method

$$\mathrm{dd}(ang) := \left| \begin{array}{l} degree \leftarrow \mathrm{floor}(ang) \\ mins \leftarrow (ang - degree)\cdot 100.0 \\ minutes \leftarrow \mathrm{floor}(mins) \\ seconds \leftarrow (mins - minutes)\cdot 100.0 \\ degree + \dfrac{minutes}{60.0} + \dfrac{seconds}{3600.0} \end{array} \right.$$

$$\mathrm{radians}(ang) := \left| \begin{array}{l} d \leftarrow \mathrm{dd}(ang) \\ d\cdot \dfrac{\pi}{180.0} \end{array} \right.$$

$$\mathrm{dms}(ang) := \left| \begin{array}{l} degree \leftarrow \mathrm{floor}(ang) \\ rem \leftarrow (ang - degree)\cdot 60 \\ mins \leftarrow \mathrm{floor}(rem) \\ rem1 \leftarrow (rem - mins) \\ secs \leftarrow rem1\cdot 60.0 \\ degree + \dfrac{mins}{100} + \dfrac{secs}{10000} \end{array} \right.$$

$$trad := \frac{\pi}{180} \qquad tdeg := \frac{180}{\pi}$$

Given

$X_A := 1000.00$ $\qquad$ $Y_A := 5300.00$

$X_B := 3100.00$ $\qquad$ $Y_B := 5000.00$

$X_C := 2200.00$ $\qquad$ $Y_C := 6300.00$

$\alpha := 109.3045$ $\qquad$ $\beta := 115.0520$

Solution - Find the coordinates of point P using the Kaestner-Burkhardt Method. Begin by computing the azimuths and distances between the known points.

$$Az_{AC} := \left[\operatorname{atan2}\left[\left(Y_C - Y_A\right),\left(X_C - X_A\right)\right]\right] \qquad \operatorname{dms}\left[\left(Az_{AC}\right)\cdot(\mathrm{tdeg})\right] = 50.11399$$

$$Az := \left[\operatorname{atan2}\left[\left(Y_C - Y_B\right),\left(X_C - X_B\right)\right]\right] \qquad Az = -0.60554$$

$$Az_{BC} := Az + (2\cdot\pi) \qquad \operatorname{dms}\left[\left(Az_{BC}\right)\cdot \mathrm{tdeg}\right] = 325.18174$$

$$a := \sqrt{\left(X_C - X_A\right)^2 + \left(Y_C - Y_A\right)^2} \qquad a = 1562.04994$$

$$b := \sqrt{\left(X_C - X_B\right)^2 + \left(Y_C - Y_B\right)^2} \qquad b = 1581.13883$$

The angle at point C is computed as are the auxiliary angles

$$\gamma := \left(Az_{AC} - Az_{BC}\right)\cdot(\mathrm{tdeg}) + 360 \qquad \operatorname{dms}(\gamma) = 84.53225$$

$$\delta_1 := 180 - \left(\frac{1}{2}\right)\cdot\left(\operatorname{dd}(\alpha) + \operatorname{dd}(\beta) + \gamma\right) \qquad \operatorname{dms}\left(\delta_1\right) = 25.15163$$

$$\lambda_0 := \left(\frac{b}{a}\right)\cdot\left[\frac{\sin\left[\operatorname{radians}\left[((\alpha))\right]\right]}{\sin\left[\operatorname{radians}((\beta))\right]}\right] \qquad \lambda_0 = 1.053482162$$

$$\lambda := \left[\mathrm{tdeg}\left[\operatorname{atan}\left[\frac{1}{\left(\lambda_0\right)}\right]\right]\right] \qquad \operatorname{dms}(\lambda) = 43.30291$$

Note that λ has an uncertainty of 180 degrees

$$\delta_2 := \left[\operatorname{atan}\left[\left(\tan\left(\operatorname{radians}\left(\operatorname{dms}\left(\delta_1\right)\right)\right)\right)\cdot\left(\left(\frac{1}{\tan\left(\operatorname{radians}\left(\operatorname{dms}(45+\lambda)\right)\right)}\right)\right)\right]\right]\cdot \mathrm{tdeg}$$

$$\operatorname{dms}\left(\delta_2\right) = 0.4214$$

$$\phi := \delta_1 + \delta_2 \qquad \operatorname{dms}(\phi) = 25.57303$$

$\theta := \delta_1 - \delta_2$ $\mathrm{dms}(\theta) = 24.33022$

Compute the distances between the point P and control points A and B

$c_1 := a\cdot\dfrac{\sin[\mathrm{radians}(\alpha) + (\phi\cdot\mathrm{trad})]}{\sin(\mathrm{radians}(\alpha))}$ $c_1 = 1162.1655$

$c_2 := b\cdot\dfrac{\sin[\mathrm{radians}(\beta) + (\theta\cdot\mathrm{trad})]}{\sin(\mathrm{radians}(\beta))}$ $c_2 = 1130.60883$

The azimuths between the control points A and B are now determined

$Az_{AP} := Az_{AC} + \phi\cdot\mathrm{trad}$ $\mathrm{dms}(Az_{AP}\cdot\mathrm{tdeg}) = 76.09102$

$Az_{BP} := Az_{BC} - \theta\cdot\mathrm{trad}$ $\mathrm{dms}(Az_{BP}\cdot\mathrm{tdeg}) = 300.45152$

Finally, the coordinates of the unknown point are computed from both points for a check

$X_P := X_A + c_1\cdot\sin(Az_{AP})$ $\boxed{X_P = 2128.390}$

$Y_P := Y_A + c_1\cdot\cos(Az_{AP})$ $\boxed{Y_P = 5578.144}$

Check

$X_P := X_B + c_2\cdot\sin(Az_{BP})$ $\boxed{X_P = 2128.390}$

$Y_P := Y_B + c_2\cdot\cos(Az_{BP})$ $\boxed{Y_P = 5578.144}$

Allan et. Al. [1968] present a slightly different approach called the Pothonot-Snellius method. Recall that the distance from C to P was designated as s and was expressed as $\dfrac{a\sin\phi}{\sin\alpha} = s = \dfrac{b\sin\theta}{\sin\beta}$. From this there are two methods of solving this problem. The first method is basically that already presented above. The second method is described as follows. Write the ratio of φ to θ by a constant K as:

$$K = \frac{\sin\theta}{\sin\varphi} = \frac{\sin(S-\varphi)}{\sin\varphi} = \frac{\sin S\cos\varphi - \cos S\sin\varphi}{\sin\varphi} = \frac{\sin S\cos\varphi}{\sin\varphi} - \cos S$$

where $S = 360^{\circ} - (\alpha+\beta+\gamma)$. This relationship is based on the fact that the sum of the interior angles in polygon ACBPA must equal 360°. Thus, one can write from this basic relationship (refer to figure 1): $\theta = \left[360^{\circ} - (\alpha+\phi+\gamma)\right] - \varphi = S - \varphi$. S represents the known angles. Manipulation of this last relationship yields

$$K + \cos S = \frac{\sin S \cos \varphi}{\sin \varphi} = \sin S \cot \varphi$$

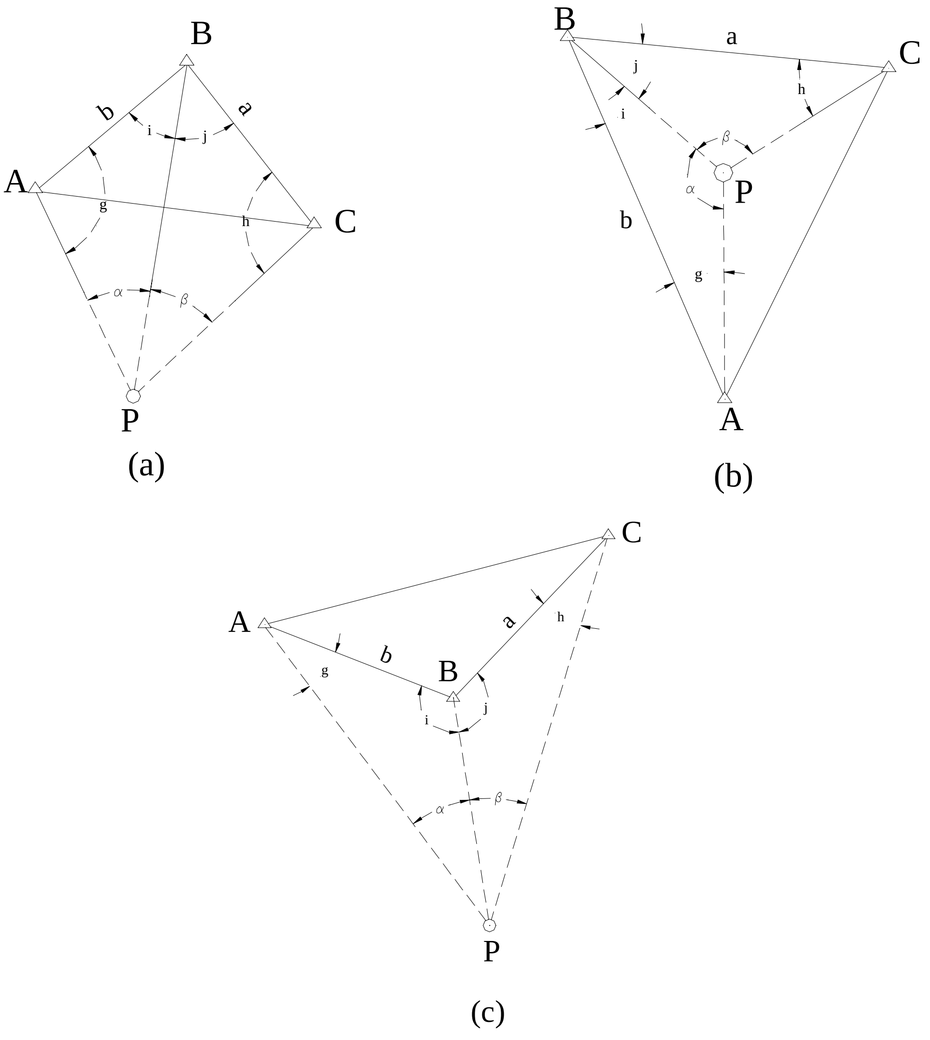
From which,

$$\cot \varphi = \frac{K + \cos S}{\sin S}$$

Solve for φ and then compute c_1 and the azimuth to determine the coordinates of point P. Alternatively, use line-line intersection to find the coordinates of the unknown point.

Another modification of the Kaestner-Burkhardt Method is that reported by the United States Coast and Geodetic Survey (USC&GS, now the National Geodetic Survey, NGS) [Hodgson, 1957; Reynolds, 1934]. Figure 2 identifies three cases of the three point resection problem. This is a modification of the USC&GS method presented in Kissam (1981) and with a slight modification in Anderson and Mikhail (1998).

The solution can be broken down into a few steps, given here without derivation.



(a) (b) (c)

Figure 2. Three scenarios for the three-point resection problem.

(a) Compute $\left(g+h\right)=360^{o}-\left(\alpha+\beta+i+j\right)$ if the problem is the same as that indicated in figure 2(a) and (b). For the configuration depicted in figure 2(c), $\left(g+h\right)=\left(i+j\right)-\left(\alpha+\beta\right)$.

(b) Then, define,

$$\cot\left(45^{o}+\theta\right)=\frac{\frac{a\sin\alpha}{b\sin\beta}-1}{\frac{a\sin\alpha}{b\sin\beta}+1}$$

where,

$$\theta=\tan^{-1}\left(\frac{b\sin\beta}{a\sin\alpha}\right)$$

(c) Further,

$$\tan\frac{1}{2}\left(g-h\right)=\cot\left(45^{o}+\theta\right)\tan\frac{1}{2}\left(g+h\right)$$

(d) Then,

$$g=\frac{\left(g+h\right)+\left(g-h\right)}{2}\qquad\text{and}\qquad h=\frac{\left(g+h\right)-\left(g-h\right)}{2}$$

(e) Finally,

$$i=180^{o}-\left(g+\alpha\right)\qquad\text{and}\qquad j=180^{o}-\left(h+\beta\right)$$

Now that all of the angles are known, the lengths of the different legs of the triangles can be found using the sine law.

From the previous example, we can see that this follows the Case 2 situation shown in figure 2. For this example we will renumber the points so that they coincide with the figure for Case 2. Thus, from the original example, point C is now designated as point B and the original B coordinate is now C. Therefore, the coordinates are:

X_A = 1,000.00	Y_A = 5,300.00
X_B = 2,200.00	Y_B = 6,300.00
X_C = 3,100.00	Y_C = 5,000.00
α = 109° 30' 45"	β = 115° 05' 20"

It was already shown that the azimuths are

COLLINS METHOD

The Collins (or Bessel's) method [Blachut et al, 1979, Faig, 1972, Klinkenberg, 1955, Zeimann, 1974] is different in that the problem is broken down into two intersections. A circle is drawn through two control points and the occupied point (as A, B, and P in figure 3). The line from P to C is extended until it intersects the circle at a point labeled H. This point is called the Collins' Auxiliary Point.

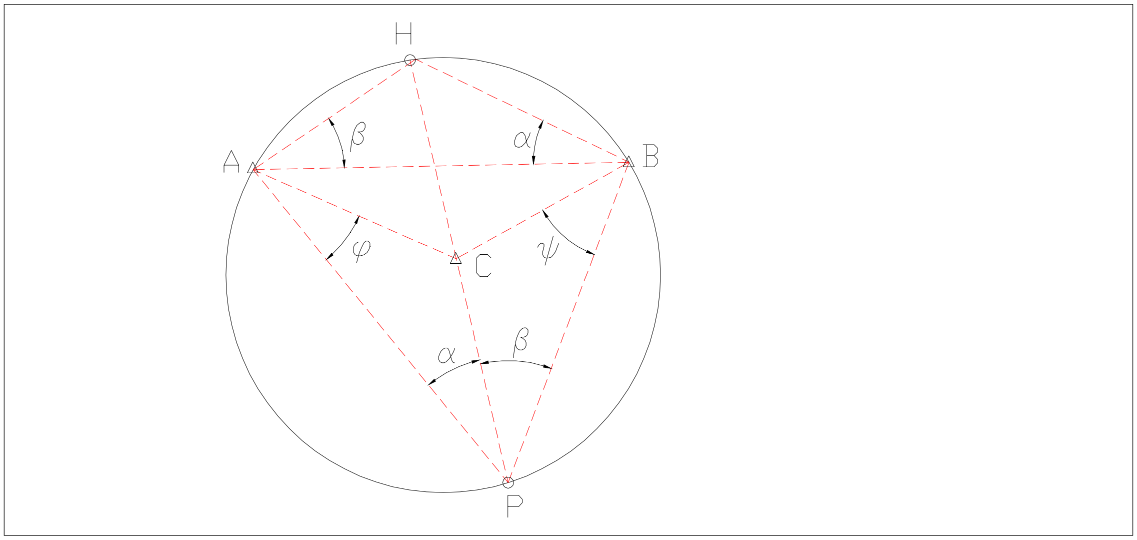


Figure 3. Three point resection problem using the Collins method.

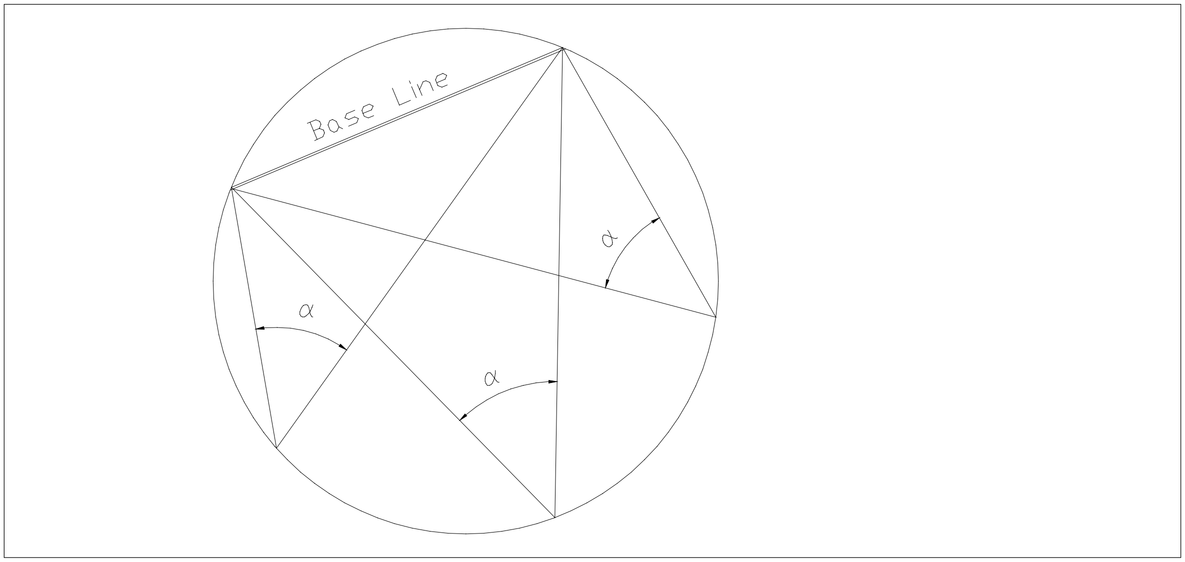


Figure 4. Geometry of circle showing that an angle on the circle subtending a base line is equal.

From the geometry of a circle, shown in figure 4, one can state that the angle formed at a point on the circumference of a circle subtending a base line on the circle is the same anywhere on the circle, provided that it is always on the same side of the base line. This property is exploited in the Collins' Method.

The solution involves five distinct steps:

1. Compute the coordinate of the Collins' Auxiliary Point, H, by intersection from both control points A and B.
2. Compute the azimuth Az_{HC} which will also yield the azimuth between C and P since $Az_{HC} = Az_{CP}$.
3. Compute the azimuth of the lines AP and BP

$$Az_{AP} = Az_{CP} - \alpha$$

$$Az_{BP} = Az_{CP} + \beta$$

4. The coordinates can be computed by intersection from A and C and also from B and C.
5. If desired, the solution can be performed using the auxiliary angles φ and ψ.

$$\varphi = Az_{AP} - Az_{AC}$$

$$\psi = Az_{BC} - Az_{BP}$$

Then, using the sine law,

$$D_{AP} = \frac{D_{AC} \sin(\alpha + \varphi)}{\sin \alpha}$$

$$D_{BP} = \frac{D_{BC} \sin(\beta + \psi)}{\sin \beta}$$

This gives

$$X_P = X_A + D_{AP} \sin Az_{AP} = X_B + D_{BP} \cos Az_{BP}$$

$$Y_P = Y_A + D_{AP} \cos Az_{AP} = Y_B + D_{BO} \cos Az_{BP}$$

Following is a MathCAD program that solves the same problem as presented earlier but this time using the Collins method.

Three Point Resection Problem
Collins Method

See the same functions as defined in the Kaestner-Burkhardt MathCAD program.

Given

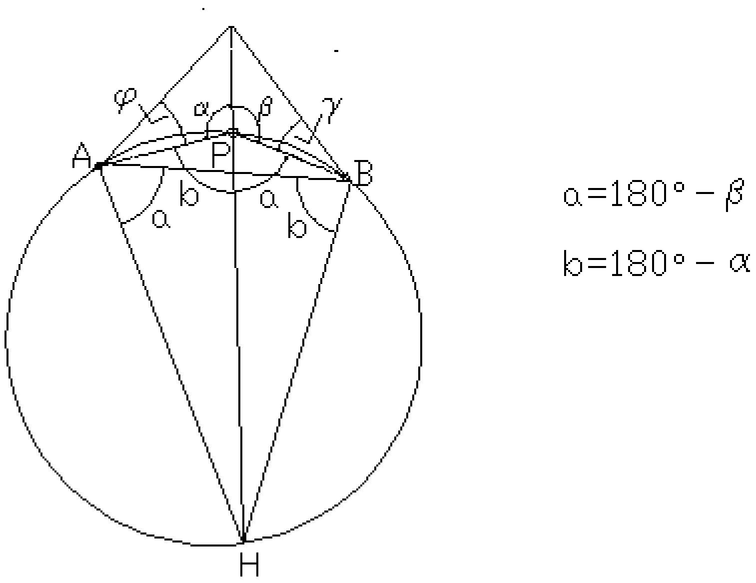
$X_A := 1000.00$ $Y_A := 5300.00$

$X_B := 3100.00$ $Y_B := 5000.00$

$X_C := 2200.00$ $Y_C := 6300.00$

$\alpha := 109.3045$ $\beta := 115.0520$

Solution - Find the coordinates of point P using the Collins Method. Begin by looking at the triangle ABH Angles are designated by the variable "a" with subscript showing backsight, station, and foresight lettering.



$a_{BAH} := 180 - dd(\beta)$ $dms(a_{BAH}) = 64.5440$

$a_{ABH} := 180 - dd(\alpha)$ $dms(a_{ABH}) = 70.2915$

$D_{AB} := \sqrt{(X_B - X_A)^2 + (Y_B - Y_A)^2}$ $D_{AB} = 2121.32034$

$Az_{AB} := atan2(Y_B - Y_A, X_B - X_A)$ $dms(Az_{AB} \cdot tdeg) = 98.07484$

$a_{AHB} := 180 - [(180 - dd(\beta)) + (180 - dd(\alpha))]$ $dms(a_{AHB}) = 44.3605$

$Az_{AH} := Az_{AB} + [\mathrm{trad}\cdot(180 - \mathrm{dd}(\beta))]$ $\mathrm{dms}(Az_{AH}\cdot \mathrm{tdeg}) = 163.02284$

$D_{AH} := \left(\frac{D_{AB}}{\sin(\mathrm{trad}\cdot a_{AHB})}\right)\cdot\sin[(180 - \mathrm{dd}(\alpha))\cdot\mathrm{trad}]$ $D_{AH} = 2847.58555$

$D_{BH} := \left(\frac{D_{AB}}{\sin(a_{AHB}\cdot\mathrm{trad})}\right)\cdot\sin[(180 - \mathrm{dd}(\beta))\cdot\mathrm{trad}]$ $D_{BH} = 2736.05413$

$X_H := X_A + D_{AH}\cdot\sin(Az_{AH})$ $X_H = 1830.59443$

$Y_H := Y_A + D_{AH}\cdot\cos(Az_{AH})$ $Y_H = 2576.24223$

$Az := \mathrm{atan2}(Y_H - Y_C, X_H - X_C)$

$Az_{CH} := \mathrm{if}(Az > 0, Az, Az + 2\cdot\pi)$ $\mathrm{dms}(Az_{CH}\cdot\mathrm{tdeg}) = 185.39552$

$Az := \mathrm{atan2}(Y_A - Y_C, X_A - X_C)$

$Az_{CA} := \mathrm{if}(Az > 0, Az, Az + 2\cdot\pi)$ $\mathrm{dms}(Az_{CA}\cdot\mathrm{tdeg}) = 230.11399$

$a_{ACP} := Az_{CA} - Az_{CH}$ $\mathrm{dms}(a_{ACP}\cdot\mathrm{tdeg}) = 44.31447$

$\phi := 180 - (\mathrm{dd}(\alpha) + a_{ACP}\cdot\mathrm{tdeg})$ $\mathrm{dms}(\phi) = 25.57303$

$Az_{AP} := (Az_{CA} - \pi) + \phi\cdot\mathrm{trad}$ $\mathrm{dms}(Az_{AP}\cdot\mathrm{tdeg}) = 76.09102$

$D_{AC} := \sqrt{(X_C - X_A)^2 + (Y_C - Y_A)^2}$ $D_{AC} = 1562.04994$

From the sine law:

$D_{AP} := \left(\frac{D_{AC}}{\sin(\mathrm{dd}(\alpha)\cdot\mathrm{trad})}\right)\cdot\sin(a_{ACP})$ $D_{AP} = 1162.1655$

$X_P := X_A + D_{AP}\cdot\sin(Az_{AP})$ $\boxed{X_P = 2128.390}$

$Y_P := Y_A + D_{AP}\cdot\cos(Az_{AP})$ $\boxed{Y_P = 5578.144}$

For a check, compute the coordinates from point B by solving for the elements in triangle BCP.

CASSINI METHOD

The Cassini approach [Blachut et al, 1979, Faig, 1972, Klinkenberg, 1955, Ziemann, 1974] to the solution of the three-point resection problem is a geometric approach. It breaks the problem down to an intersection of two circles where one of the intersection points is the unknown point P while the other is one of the three control points. This is depicted in figure 5.

The solution is shown as follows:

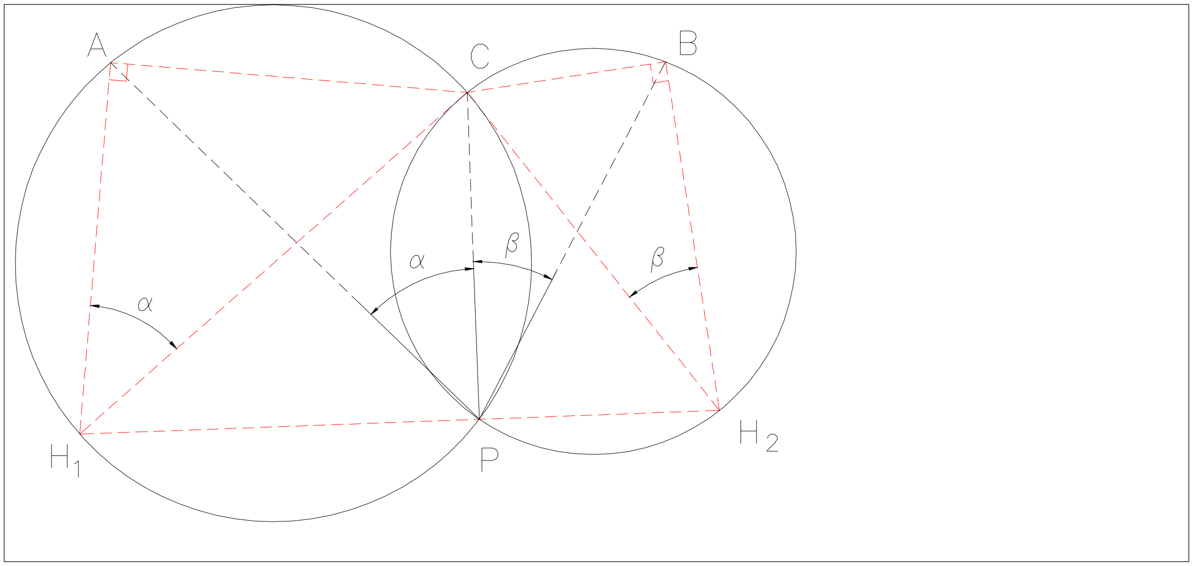


Figure 5. Three point resection problem as proposed by Cassini.

Compute the coordinates of the auxiliary points H_1 and H_2. First the azimuths between A and H_1 and B and H_1 are determined.

$$\mathrm{Az}_{\mathrm{AH}_1} = \mathrm{Az}_{\mathrm{AC}} + 90^{\circ}$$

$$\mathrm{Az}_{\mathrm{BH}_1} = \mathrm{Az}_{\mathrm{BC}} - 90^{\circ}$$

From triangle ACH_1, the distance from A to H_1 can be computed.

$$\tan\alpha = \frac{D_{\mathrm{AC}}}{D_{\mathrm{AH}_1}}$$

$$D_{\mathrm{AH}_1} = \frac{D_{\mathrm{AC}}}{\tan\alpha} = \frac{X_C - X_A}{\sin \mathrm{Az}_{\mathrm{AC}} \tan\alpha} = \frac{Y_C - Y_A}{\cos \mathrm{Az}_{\mathrm{AC}} \tan\alpha}$$

Since the angle at A is 90^o,

$$\sin Az_{AH_1} = \cos Az_{ac} \qquad ; \qquad \cos Az_{AH_1} = -\sin Az_{AC}$$

Then,

$$X_{H_1} = X_A + D_{AH_1} \sin Az_{AH_1} = X_A + (Y_C - Y_A)\cot\alpha$$

$$Y_{H_1} = Y_A + D_{AH_1} \cos Az_{AH_1} = Y_A - (X_C - X_A)\cot\alpha$$

The coordinates for H_2 are computed in like fashion.

$$D_{BH_2} = \frac{D_{BC}}{\tan\beta} = \frac{X_C - X_B}{\sin Az_{bc} \tan\beta} = \frac{Y_C - Y_B}{\cos Az_{bc} \tan\beta}$$

$$\sin Az_{BH_2} = -\cos Az_{BC} \qquad ; \qquad \cos Az_{BH_2} = \sin Az_{bc}$$

$$X_{H_2} = X_B + D_{BH_2} \sin Az_{bh_2} = X_B - (Y_C - Y_B)\cot\beta$$

$$Y_{H_2} = Y_B + D_{BH_2} \cos Az_{BH_2} = Y_B - (X_C - X_B)\cot\beta$$

An alternative approach to coming up with the formulas for X_H and Y_H can also be presented. This approach breaks the solution of the Cassini Method down to 5 equations. From the equation of the intersections of two lines, we can write:

$$X_C - X_B = (Y_C - Y_B)\tan Az_{BC}$$

This can also be written as

$$X_C - X_B = (Y_C - Y_A)\tan Az_{BC} + (Y_A - Y_B)\tan Az_{bc}$$

But,

$$X_C - X_A = (Y_C - Y_A)\tan Az_{AC}$$

Solving these last two equations can be done by subtracting the last equation from the preceding equation resulting in

$$\begin{array}{l} \quad X_C - X_B = \left(Y_C - Y_A\right)\tan Az_{BC} + \left(Y_A - Y_B\right)\tan Az_{BC} \\ -\left[X_C - X_A = \left(Y_C - Y_A \tan Az_{AC}\right) \qquad\qquad\qquad\qquad\qquad\qquad \right] \\ \hline X_A - X_B = \left(Y_C - Y_A\right)\left(\tan Az_{BC} - \tan Az_{AC}\right) + \left(X_A - X_B\right)\tan Az_{BC} \end{array}$$

Rearranging yields

$$Y_C = Y_A + \frac{\left(X_A - X_B\right) + \left(Y_A - Y_B\right)\tan Az_{bc}}{\tan Az_{BC} - \tan Az_{AC}}$$

Using the form of this last equation, one can write express the Y-coordinate of the Cassini auxiliary point, H_1 as

$$Y_{H_1} = Y_A + \frac{\left(Y_C - Y_A\right)\tan Az_{CH_1} - \left(X_C - X_A\right)}{\tan Az_{CH_1} - \tan Az_{AH_1}}$$

But,

$$\left(Y_C - Y_A\right)\tan Az_{CA} = \left(X_C - X_A\right)$$

and

$$\tan Az_{AH_1} \tan Az_{CA} = -1$$

then the Y-coordinate for H_1 becomes, after multiplication by tan Az_{CA}

$$Y_{H_1} = Y_A + \left(X_C - X_A\right)\tan\left(Az_{CH_1} - Az_{CA}\right)$$

The X-coordinate can also be developed in a similar fashion yielding

$$X_{H_1} = X_A - \left(Y_C - Y_A\right)\tan\left(Az_{CH_1} - Az_{CA}\right)$$

But $Az_{CH_1} - Az_{CA} = -\left(90^\circ - \alpha\right)$. Then,

$$X_{H_1} = X_A + D_{AH_1}\sin Az_{AH_1} = X_A + \left(Y_C - Y_A\right)\cot\alpha$$

$$Y_{H_1} = Y_A + D_{AH_1}\cos Az_{AH_1} = Y_A - \left(X_C - X_A\right)\cot\alpha$$

The coordinates for H_2 can be developed in a similar fashion and they are given above.

Next, compute the azimuth between the two auxiliary points, H_1 and H_2.

$$\mathrm{Az}_{H_1H_2} = \tan^{-1}\left[\frac{X_{H_2} - X_{H_1}}{Y_{H_2} - Y_{H_1}}\right]$$

As before, one can write the equation of intersection containing the unknown point P as:

$$Y_P - Y_{H_1} = \frac{\left(X_{H_1} - X_C\right) + \left(Y_C - Y_{H_1}\right)\tan \mathrm{Az}_{CP}}{\tan \mathrm{Az}_{CP} - \tan \mathrm{Az}_{H_1P}}$$

or,

$$Y_P = \frac{Y_{H_1}\tan \mathrm{Az}_{CP} - Y_{H_1}\tan \mathrm{Az}_{H_1H_2} + Y_C \tan \mathrm{Az}_{CP} - Y_{H_1}\tan \mathrm{Az}_{CP} - \left(X_C - X_{H_1}\right)}{\tan \mathrm{Az}_{CP} - \tan \mathrm{Az}_{H_1P}}$$

$$= \frac{Y_C \tan \mathrm{Az}_{CP} - Y_{H_1}\tan \mathrm{Az}_{H_1P} - \left(X_C - X_{H_1}\right)}{\tan \mathrm{Az}_{CP} - \tan \mathrm{Az}_{H_1P}}$$

But,

$$Y_P = \frac{n\, Y_{H_1} + \left(1/n\right) Y_C + X_C - X_{H_1}}{N}$$

Thus,

$$\tan \mathrm{Az}_{CP} = -\frac{1}{\tan \mathrm{Az}_{H_1P}}$$

where: $n = \tan \mathrm{Az}_{H_1P}$

$N = n + \left(1/n\right)$

The X-coordinate of the unknown point can be expressed in a similar form as:

$$X_P = \frac{n\,X_C + \left(1/n\right)X_{H_1} + Y_C - Y_{H_1}}{N}$$

The same problem used in the previous methods follows showing the application of the Cassini method to solving the resection problem.

Three Point Resection Problem
Cassini Method

See the same functions as defined in the Kaestner-Burkhardt MathCAD program.

Given

$X_A := 1000.00$ $Y_A := 5300.00$

$X_B := 3100.00$ $Y_B := 5000.00$

$X_C := 2200.00$ $Y_C := 6300.00$

$\alpha := 109.3045$ $\beta := 115.0520$

Solution - Find the coordinates of point P using the Cassini Method.

$X_{H1} := X_A + \left(Y_C - Y_A\right)\cot\left(dd\left(\alpha\right)\cdot trad\right)$ $X_{H1} = 645.63588$

$Y_{H1} := Y_A + \left(X_A - X_C\right)\cdot\cot\left(dd\left(\alpha\right)\cdot trad\right)$ $Y_{H1} = 5725.23694$

$X_{H2} := X_B + \left(Y_B - Y_C\right)\cot\left(dd\left(\beta\right)\cdot trad\right)$ $X_{H2} = 3708.6571$

$Y_{H2} := Y_B + \left(X_C - X_B\right)\cdot\cot\left(dd\left(\beta\right)\cdot trad\right)$ $Y_{H2} = 5421.378$

$Az_{H1H2} := \operatorname{atan2}\left(Y_{H2} - Y_{H1}, X_{H2} - X_{H1}\right)$ $dms\left(Az_{H1H2}\cdot tdeg\right) = 95.39552$

$n := \tan\left(Az_{H1H2}\right)$

$N := n + \frac{1}{n}$

$$Y_P := \frac{\left[n\cdot Y_{H1} + \left(\frac{1}{n}\right)\cdot Y_C + X_C - X_{H1}\right]}{N}$$

$\boxed{Y_P = 5578.14421}$

$$X_P := \frac{n \cdot X_C + \left(\frac{1}{n}\right) \cdot X_{H1} + Y_C - Y_{H1}}{N}$$

$X_P = 2128.3902$

TIENSTRA METHOD

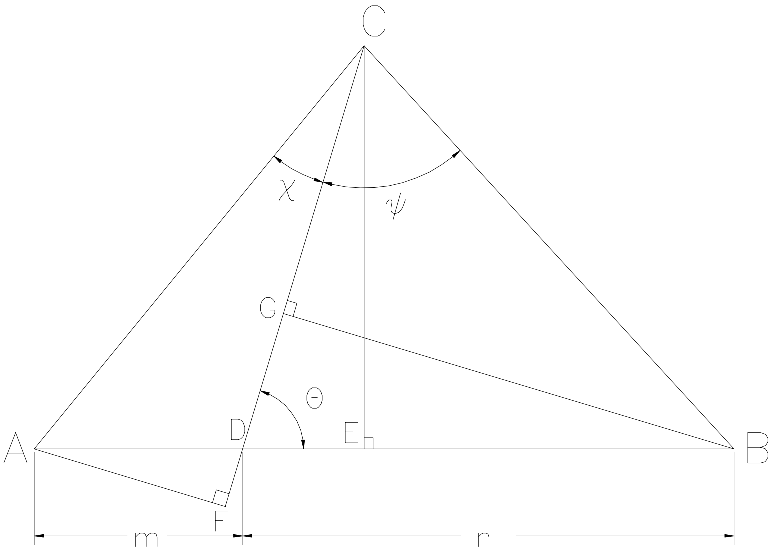


Figure 6. Basic geometry outlining the principles of the Tienstra Method.

The Tienstra method [see Bannister et al, 1984] is also referred to as the Barycentric method. An easy to understand proof is given in Allan et al [1968]. Figure 6 shows a triangle formed from the known control points. Line CD divides the angle at C into two components: χ and ψ. Line AB is also divided into two components: m and n. The angle θ is formed by the intersection of the line CD with the line AB. From figure 6 one can also see that line CE is perpendicular to line AB. Thus,

$$\cot \angle_A = \frac{D_{AE}}{D_{CE}}$$

$$\cot \angle_B = \frac{D_{EB}}{D_{CE}}$$

$$\cot \theta = \frac{D_{DE}}{D_{CE}}$$

Then,

$$\frac{m}{n}=\frac{D_{AD}}{D_{DB}}=\frac{D_{AE}-D_{DE}}{D_{DE}+D_{EB}}=\frac{D_{CE}\left(\cot\angle_{A}-\cot\theta\right)}{D_{CE}\left(\cot\angle_{B}-\cot\theta\right)}$$

which upon further manipulation yields

$$\frac{m}{n}=\frac{\cot\angle_{A}-\cot\theta}{\cot\angle_{B}+\cot\theta}$$

$$m\cot\angle_{B}+m\cot\theta=n\cot\angle_{A}-n\cot\theta$$

or

$$\left(m+n\right)\cot\theta=n\cot\angle_{A}-m\cot\angle_{B}$$

Since lines AF and BG are perpendicular to line CF, one can write

$$\cot\chi=\frac{D_{CF}}{D_{AF}}\qquad\Rightarrow\qquad D_{AF}=\frac{D_{CF}}{\cot\chi}$$

$$\begin{aligned}\cot\theta&=\frac{D_{DF}}{D_{AF}}\qquad\Rightarrow\qquad D_{AF}=\frac{D_{DF}}{\cot\theta}\\&=\frac{D_{GD}}{D_{BG}}\qquad\Rightarrow\qquad D_{BG}=\frac{D_{GD}}{\cot\theta}\end{aligned}$$

$$\cot\psi=\frac{D_{CG}}{D_{BG}}\qquad\Rightarrow\qquad D_{BG}=\frac{D_{CG}}{\cot\psi}$$

From these relationships, equate D_{AF}

$$\frac{D_{CF}}{\cot\chi}=\frac{D_{DF}}{\cot\theta}\qquad\Rightarrow\qquad\frac{D_{CF}}{D_{DF}}=\frac{\cot\chi}{\cot\theta}$$

and equating the distance D_{BG}

$$\frac{D_{GD}}{\cot\theta}=\frac{D_{CG}}{\cot\psi}\qquad\Rightarrow\qquad\frac{D_{GD}}{D_{CF}}=\frac{\cot\theta}{\cot\psi}$$

From figure 6 we can also write

$$D_{CD} = D_{CF} - D_{DF} = \frac{D_{DF} \cot\chi}{\cot\theta} - D_{DF}$$

$$D_{CF} - D_{DF} = D_{DF}\left(\frac{\cot\chi}{\cot\theta} - 1\right)$$

$$\frac{D_{CF} - D_{DF}}{D_{DF}} = \frac{\cot\chi - \cot\theta}{\cot\theta}$$

Also, we have,

$$D_{CD} = D_{CG} + D_{DG} = \frac{D_{DG} \cot\psi}{\cot\theta} + D_{DG}$$

$$= D_{DG}\left(\frac{\cot\psi}{\cot\theta} + 1\right)$$

$$\frac{D_{CG} + D_{DG}}{D_{DG}} = \frac{\cot\psi + \cot\theta}{\cot\theta}$$

From above one can see that the distance from C to D can be expressed as

$$D_{CD} = D_{DF}\left(\frac{\cot\chi}{\cot\theta} - 1\right)$$

But from figure 6 we can write the following two relationships

$$\cos\theta = \frac{D_{DF}}{D_{AD}} = \frac{D_{DF}}{m} \qquad \Rightarrow \qquad D_{DF} = m\cos\theta$$

$$\cos\theta = \frac{D_{DG}}{n} \qquad \Rightarrow \qquad D_{DG} = n\cos\theta$$

Substitute these values for D_{DF} and D_{DG} into the relationships derived above. This is shown as:

$$D_{DG} = D_{DG}\left(\frac{\cot\psi}{\cot\theta}+1\right) \qquad\qquad D_{CD} = D_{DF}\left(\frac{\cot\chi}{\cot\theta}-1\right)$$

$$= n\cos\theta\left(\frac{\cot\psi}{\cot\theta}+1\right) \qquad\qquad = m\cos\theta\left(\frac{\cot\chi}{\cot\theta}-1\right)$$

Equating the two values for D_{CD} yields

$$n\cos\theta\left(\frac{\cot\psi+\cot\theta}{\cot\theta}\right) = m\cos\theta\left(\frac{\cot\chi-\cot\theta}{\cot\theta}\right)$$

$$n\cot\psi + n\cot\theta = m\cot\psi - m\cot\theta$$

$$(m+n)\cot\theta = m\cot\chi - n\cot\psi$$

The three-point resection problem is shown in figure 7. Point P is the occupied point and points A, B, and C are the control points that are observed. The measured angles are α, β, and γ. The other angles are numbered in a clockwise manner from point A. Recall that from the intersection problem, the coordinates of a point, such as point C, can be computed as:

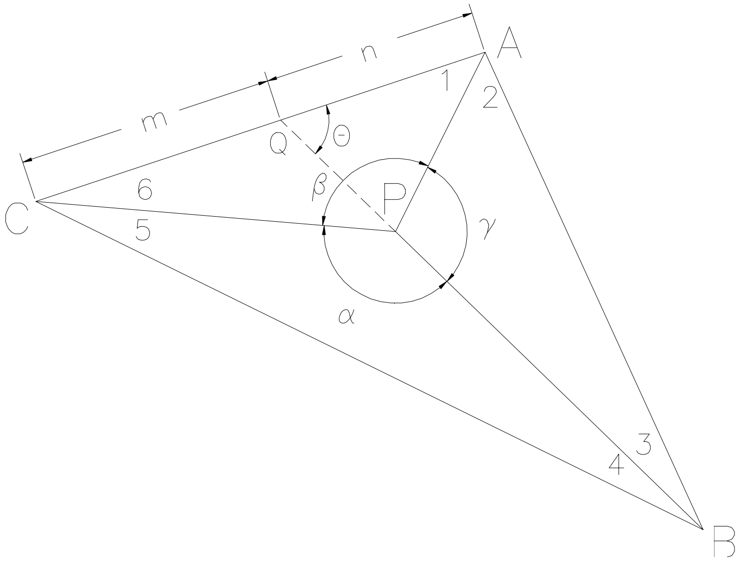


Figure 7. Three point resection problem using the Tienstra Method.

$$X_C = \frac{(Y_A - Y_B) + X_A \cot\beta + X_B \cot\alpha}{\cot\alpha + \cot\beta}$$

$$X_C \ (\cot\alpha + \cot\beta) = X_A \cot\beta + X_B \cot\alpha + Y_A - Y_B$$

where α is the angle at A and β is the angle at B. Using this basic relationship, the X-coordinate at point P can be computed as follows:

Adding these three equations yields:

$$\begin{aligned} X_P \ (\cot 1 + \cot 2 + \cot 3 + \cot 4 + \cot 5 + \cot 6) &= X_A(\cot 3 + \cot 6) \\ &+ X_B(\cot 2 + \cot 5) + X_C(\cot 4 + \cot 1) \end{aligned}$$

This is usually represented as

$$X_P(L_1 + L_2 + L_3) = L_1X_A + L_2X_B + L_3X_C$$

where: L_1 = cot 3 + cot 6
L_2 = cot 2 + cot 5
L_3 = cot 4 + cot 1

The X-coordinate is computed as

$$X_P = \frac{L_1X_A + L_2X_B + L_3X_C}{L_1 + L_2 + L_3}$$

$$X_P \ (\cot 2 + \cot 3) = X_A \cot 3 + X_B \cot 2 + Y_A - Y_B$$
$$X_P \ (\cot 4 + \cot 5) = X_B \cot 5 + X_C \cot 4 + Y_B - Y_C$$
$$X_P \ (\cot 1 + \cot 6) = X_C \cot 1 + X_A \cot 6 + Y_C - Y_A$$

In a similar fashion, the Y-coordinate can be written, from the intersection problem

$$Y_C = \frac{(X_B - X_A) + Y_A \cot\beta + Y_B \cot\alpha}{\cot\alpha + \cot\beta}$$

which can be shown, after the same manipulation performed on the X-coordinate, as

$$Y_P = \frac{L_1Y_A + L_2Y_B + L_3Y_C}{L_1 + L_2 + L_3}$$

From figure 7, the line BP was extended until it intersected the line AC at a point labeled Q. This divides the line into two parts: m and n. Recall that the angle $\angle_{CPQ} = 180^O - \alpha$ and $\angle_{APQ} = 180^o - \gamma$. Recall that we wrote earlier: $(m+n)\cot\theta = m\cot\chi - n\cot\psi$. Using the geometry from figure7, this becomes,

$$(m+n)\cot\theta = m\cot 4 - n\cot 3$$

Recall that earlier we wrote the relationship: $(m+n)\cot\theta = m\cot\angle_A - n\cot\angle_B$ which can be written as (considering the geometry in figure 7)

$$(m+n)\cot\theta = n\cot 6 - m\cot 1$$

Equating these last two formulas yields the following formula,

$$m(\cot 4 + \cot 1) = n(\cot 3 + \cot 6)$$

Using $(m+n)\cot\theta = m\cot\chi - n\cot\psi$ and $(m+n)\cot\theta = m\cot\angle_A - n\cot\angle_B$ again, write

$$(m+n)\cot\theta = -m\cot\alpha + n\cot\gamma$$
$$(m+n)\cot\theta = n\cot\angle_C - m\cot\angle_A$$

Equating these last two equations gives

$$m(\cot\angle_A - \cot\alpha) = n(\cot\angle_C - \cot\gamma)$$

Using this formula, equate it with $m(\cot 4 + \cot 1) = n(\cot 3 + \cot 6)$ giving us the next equation

$$\frac{\cot 1 + \cot 4}{\cot 3 + \cot 6} = \frac{n}{m} = \frac{\cot\angle_A - \cot\alpha}{\cot\angle_C - \cot\gamma}$$

or

$$\frac{L_3}{L_1} = \frac{K_3}{K_1}$$

$$1/K_1 = \cot\angle_A - \cot\alpha$$

where: $1/K_2 = \cot\angle_B - \cot\beta$

$$1/K_3 = \cot\angle_C - \cot\gamma$$

In a similar fashion, one can easily show that

$$\frac{L_3}{L_2} = \frac{K_3}{K_2}$$

Therefore,

$$\frac{L_1}{K_1} = \frac{L_2}{K_2} = \frac{L_3}{K_3} = W$$

from which,

$$L_1 = K_1 W$$
$$L_2 = K_2 W$$
$$L_3 = K_3 W$$

and

$$L_1 + L_2 + L_3 = W\left(K_1 + K_2 + K_3\right)$$

Thus,

$$\frac{L_1}{L_1 + L_2 + L_3} = \frac{K_1}{K_1 + K_2 + K_3}$$

$$\frac{L_2}{L_1 + L_2 + L_3} = \frac{K_2}{K_1 + K_2 + K_3}$$

$$\frac{L_3}{L_1 + L_2 + L_3} = \frac{K_3}{K_1 + K_2 + K_3}$$

Substituting these relationships back into the equations for X_p and Y_P which were expressed in terms of L_1, L_2, and L_3 that were presented earlier yields the final form for computing the coordinates using the Tienstra method.

$$X_P = \frac{K_1 X_A + K_2 X_B + K_3 X_C}{K_1 + K_2 + K_3}$$

$$Y_P = \frac{K_1 Y_A + K_2 Y_B + K_3 Y_C}{K_1 + K_2 + K_3}$$

An example using MathCAD follows:

Three Point Resection Problem
Tienstra Method

See the same functions as defined in the Kaestner-Burkhardt MathCAD program

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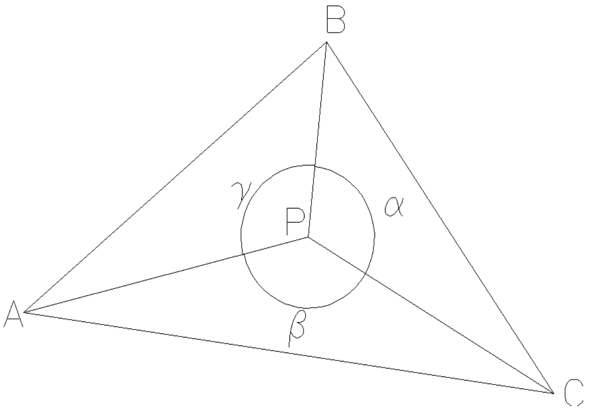
This MathCAD example is the same example used in the other methods. There is a slight difference in that the triangle is lettered in a clockwise manner and α is the clockwise angle from line PB to line PC, β is the clockwise angle from line PC to line PA, and γ is the clockwise angle from line PA to line PB. See the following figure.



Given:

$X_A := 1000.00$ $Y_A := 5300.00$ $\alpha := 115.0520$

$X_B := 2200.00$ $Y_B := 6300.00$ $\beta := 135.2355$

$X_C := 3100.00$ $Y_C := 5000.00$ $\gamma := 109.3045$

Solution - Find the coordinates of point P using the Tienstra Method.

$Az_{AB} := 50.1140$ $Az_{BA} := Az_{AB} + 180$

$Az_{CB} := 325.1817$ $Az_{BC} := Az_{CB} - 180$

$Az_{AC} := 98.0748$ $Az_{CA} := Az_{AC} + 180$

$A := dd\left(Az_{AC}\right) - dd\left(Az_{AB}\right)$ $dms(A) = 47.5608$

$B := dd\left(Az_{BA}\right) - dd\left(Az_{BC}\right)$ $dms(B) = 84.5323$

$C := dd\left(Az_{CB}\right) - dd\left(Az_{CA}\right)$ $dms(C) = 47.1029$

Place the angles into radians

$Ar := A \cdot trad$ $Br := B \cdot trad$ $Cr := C \cdot trad$

$\alpha r := radians(\alpha)$ $\beta r := radians(\beta)$ $\gamma r := radians(\gamma)$

Solve for the constants used in the Tienstra Method

$K_1 := (\cot(Ar) - \cot(\alpha r))^{-1}$ $K_1 = 0.72959$

$K_2 := (\cot(Br) - \cot(\beta r))^{-1}$ $K_2 = 0.90626$

$K_3 := (\cot(Cr) - \cot(\gamma r))^{-1}$ $K_3 = 0.78052$

The solution is:

$$X_P := \frac{(K_1 \cdot X_A + K_2 \cdot X_B + K_3 \cdot X_C)}{K_1 + K_2 + K_3} \qquad X_P = 2128.391$$

$$Y_P := \frac{(K_1 \cdot Y_A + K_2 \cdot Y_B + K_3 \cdot Y_C)}{K_1 + K_2 + K_3} \qquad Y_P = 5578.1451$$

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