6 MISSING DATA IN A POLYGON

Surveying Engineering Department
Ferris State University

COMPUTING MISSING DATA USING THE GEOMETRIC APPROACH

The conventional approach of solving the missing data problem is shown in figure 1. On the left is the traverse as it exists in the field. But, two data items are missing. Depending on the situation, a combination of two items given from s_1, s_2, α_1, or α_2 are unknown. Then, one simply draws the sides of the traverse which are known. Then the unknown lines are grouped together. Thus, the traverse consists of an open traverse with line numbers 12, 14 and 15 drawn in a group. The coordinates of points A and C' are computed by processing this open traverse. Then the distance and directions between A and C' can be computed and the triangle composed of AB'C' is computed using coordinate geometry. There are three basic forms in which the missing data problem can be formulated. An example from Hashimi [1988] is shown for each case.

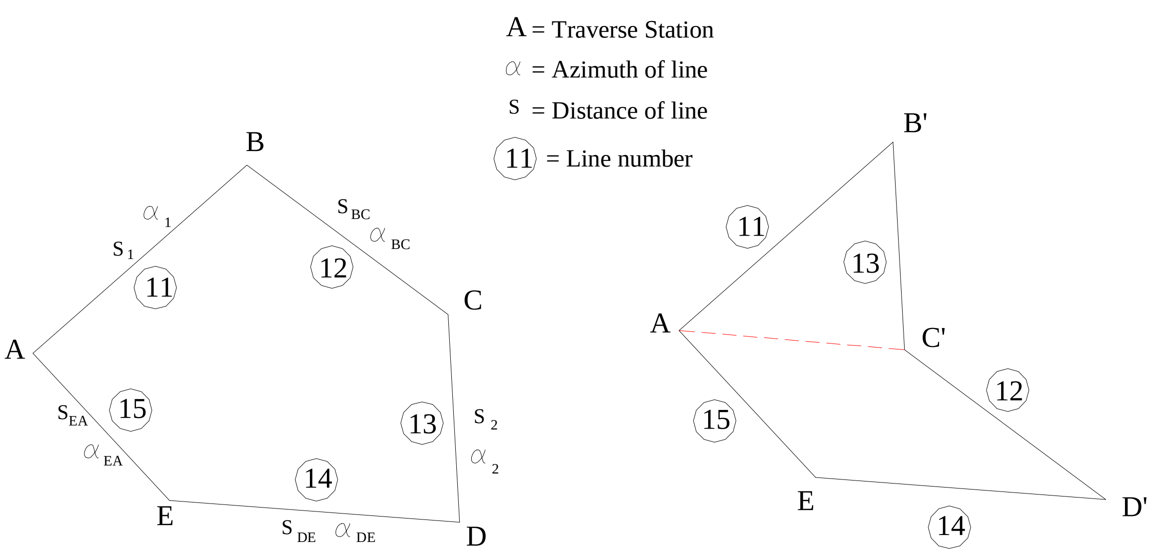


Figure 1. Traverse showing missing elements.

1. Missing the distance and direction of the same line. This problem is very simple to solve. Determine the X and Y coordinates of each of the points in the traverse. This usually means starting at one known point and going both clockwise and counterclockwise until the unknown line is reached. Once the coordinates are found, inverse between them to find the distance and direction of the line.

2. Missing two distances of two adjacent lines, or two directions of two adjacent lines or a distance and a direction of two adjacent lines. For example, look at figure 2. The data that is known is given in the accompanying table.

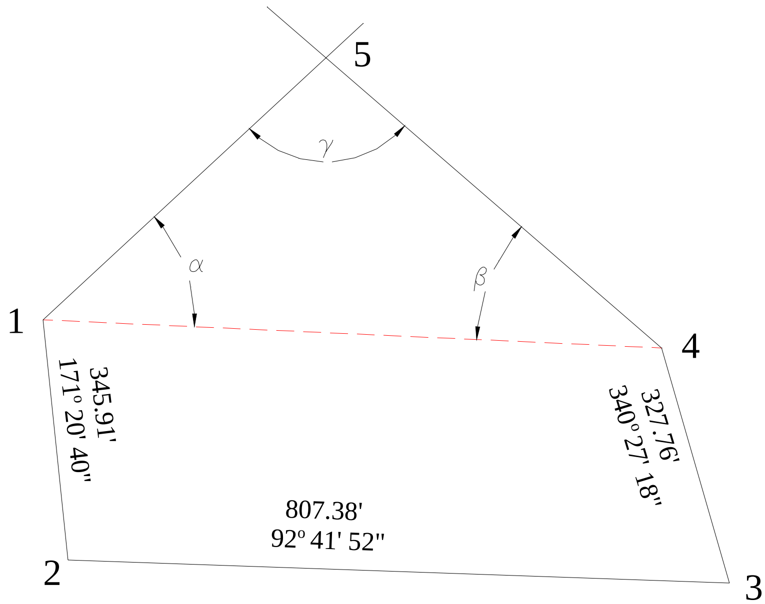


Figure 2. Example problem for computing missing data on two adjacent lines.

STA	DISTANCE	AZIMUTH	X	Y	COMMENTS
1			1000.00	500.00	Assumed
	345.91	171^{O} 20' 40"			
2			1052.06	158.03	
	807.38	92^{O} 41' 52"			
3			1858.54	120.03	
	327.76	340^{O} 27' 18"			
4			1748.89	428.90	

Compute the distance and azimuth of line 4-1.

$$D_{4-1} = \sqrt{(X_1 - X_4)^2 + (Y_1 - Y_4)^2} = \sqrt{(1000.00 - 1748.89)^2 + (500.00 - 428.90)^2}$$
$$D_{4-1} = 752.26'$$

$$Az_{4-1} = \tan^{-1}\left[\frac{X_1 - X_4}{Y_1 - Y_4}\right] = \tan^{-1}\left[\frac{(1000.00 - 1748.89)}{(500.00 - 428.90)}\right]$$
$$Az_{4-1} = 275^{o}\ 25'\ 24''$$

Then solve for the missing elements within triangle 1-4-5. For example, if we know that the azimuth between points 1 and 5 (Az_{1-5}) was 33^{o} 47' 32" and $Az_{4-5} = 314^{o}$ 51'

18”, then the problem is one of finding the distances between points 1 and 5 and between 4 and 5. Solve for the interior angles of the triangle by

$$\alpha = Az_{1-4} - Az_{1-5} = \left(95^o 25'24''\right) - \left(33^o 47'32''\right)$$
$$\alpha = 61^o\ 37'\ 52''$$

$$\beta = Az_{4-5} - Az_{4-1} = \left(314^o 51'18''\right) - \left(275^o 25'24''\right)$$
$$\beta = 39^o\ 25'\ 54''$$

$$\gamma = 180^o - \left(\alpha + \beta\right) = 180^o - \left[\left(61^o\ 37'\ 52''\right) + \left(39^o\ 25'\ 54''\right)\right]$$
$$\gamma = 78^o\ 56'\ 14''$$

Then, using the sine law, the distances can be easily computed.

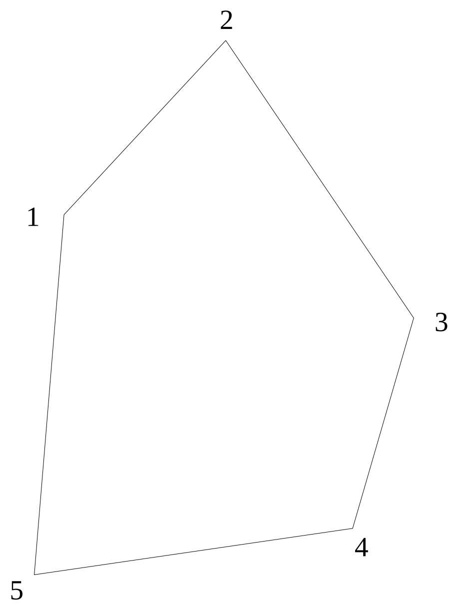
$$\frac{D_{4-5}}{\sin\alpha} = \frac{D_{1-5}}{\sin\beta} = \frac{D_{1-4}}{\sin\gamma}$$

$$D_{1-5} = \left(\frac{D_{1-4}}{\sin\gamma}\right)\sin\beta = \left(\frac{752.26}{\sin 78^o 56'14''}\right)\sin 39^o 25'54''$$
$$D_{1-5} = 486.85'$$

$$D_{4-5} = \left(\frac{D_{1-4}}{\sin\gamma}\right)\sin\alpha = \left(\frac{752.26}{\sin 78^o 56'14''}\right)\sin 61^o 37'52''$$
$$D_{4-5} = 674.45'$$

If the problem was defined in terms of the distances with the unknowns being the directions, a similar approach would be used. Here, the cosine law would be utilized to determine the unknown angles (α and β). The relationship of these angles with the azimuths of the lines between 1 and 5 and between 4 and 5 are shown above.

3. The last case involves 2 unknown distances or directions on two non-adjacent lines or an unknown distance on one line and an unknown direction on another non-adjacent line. The conventional approach is one of reconstruction of the problem. For example, Hashimi [1988] gives the following shown in figure 3 and the accompanying table.



STA	Distance	Azimuth
1		
	452.97'	37°42'17"
2		
	?	145°14'20"
3		
	567.92'	199°41'19"
4		
	602.53'	245°11'32"
5		
	?	2°21'43"
6		

Figure 3. Example of two unknown distances on two non-adjacent lines.

Starting with assumed coordinates of 1000.00 and 1000.00 for point 1, compute the coordinates of the lines which have both distance and direction known. For example, the traverse would be shown by leg 1-2, followed with leg 3-4 (with point 3 of this leg coinciding with point 2 in the first line), followed with the last leg, 4-5. Then, the new configuration looks like that shown in the figure 4.

With the coordinates now known for points A and B, compute the distance and direction between the two points,

$$D_{AB} = \sqrt{(X_B - X_A)^2 + (Y_B - Y_C)^2} = \sqrt{(1000.00 - 538.28)^2 + (1000.00 - 570.22)^2}$$
$$D_{AB} = 630.79'$$

$$Az_{AB} = \tan^{-1}\left[\frac{(X_B - X_A)}{(Y_B - Y_C)}\right] = \tan^{-1}\left[\frac{(1000.00 - 538.28)}{(1000.00 - 570.22)}\right]$$
$$Az_{AB} = 47^\circ\ 03'\ 07''$$

With the azimuth between points A and B now know, the angles (α β, and γ) can now be computed.

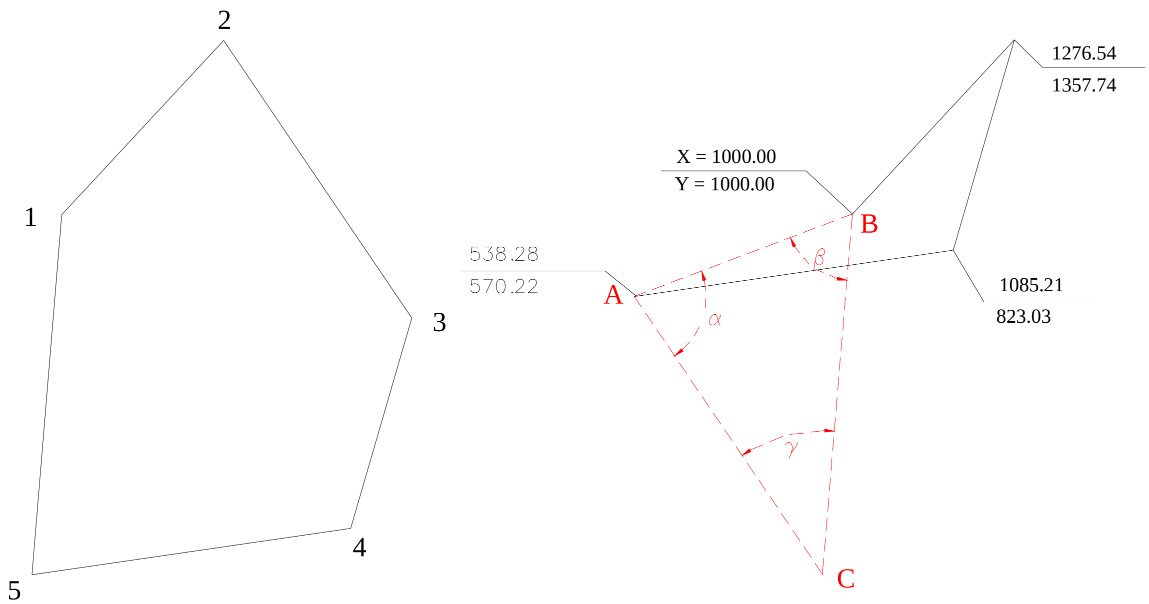


Figure 4. Construction of traverse by combining known sides into a group.

$$\alpha = Az_{AC} - Az_{AB} = (145^\circ 14'20'') - (47^\circ 03'07'') = 98^\circ\ 11'\ 13''$$

$$\beta = Az_{BA} - Az_{BC} = (227^\circ 03'07'') - (182^\circ 21'43'') = 44^\circ\ 41'\ 24''$$

$$\gamma = 360^\circ - (Az_{CA} - Az_{CB}) = 360^\circ - [(325^\circ 14'20'') - (2^\circ 21'43'')] = 37^\circ\ 07'\ 23''$$

Now that the angles are known, use the sine law to compute the corresponding distances.

$$D_{BC} = \left(\frac{D_{AB}}{\sin\gamma}\right)\sin\alpha = \left(\frac{630.79}{\sin 37^\circ 07'23''}\right)\sin 98^\circ 11'13'' = \underline{1034.52}$$

$$D_{AC} = \left(\frac{D_{AB}}{\sin\gamma}\right)\sin\beta = \left(\frac{630.79}{\sin 37^\circ 07'23''}\right)\sin 44^\circ 41'24'' = \underline{735.04}$$

Finally, check to see that the results are correct by computing the latitudes and departures for the whole traverse.

STA	DISTANCE	AZIMUTH	LATITUDE	DEPARTURE
1				
	452.17	37° 42’ 17”	+357.74	+276.54
2				
	735.04	145° 14’ 20”	-603.86	+419.09
3				
	567.92	199° 41’ 19”	-534.72	-191.34
4				
	602.53	245° 11’ 32”	-252.80	-546.92
5				
	1034.52	2° 21’ 43”	+1033.64	+42.63
1				
			0.00	0.00

As one can see, the values work.

COMPUTING MISSING DATA USING THE ANALYTICAL APPROACH

While the method of rearranging the traverse is commonly used to explain how missing data from a polygon can be computed, there are alternatives. Root [1970] suggests another approach by exploiting the condition that in a closed loop traverse the sum of the latitudes and departures must both equal zero. From this, two equations can be written with two unknowns. These equations are solved simultaneously. The unknowns can be classified into three different categories.

Category 1 is the situation where the two unknowns are both lengths of the traverse. The following two equations can be written

$$\begin{aligned} D_1 \sin Az_1 + D_2 \sin Az_2 + \Delta x = 0 \\ D_1 \cos Az_1 + D_2 \cos Az_2 + \Delta y = 0 \end{aligned} \qquad (1)$$

where D_i is the distance between two points, Az_i is the azimuth of the line, 1 and 2 indicates the lines with the unknown parameters, and Δx and Δy are the sums of the departures and latitudes of the known lengths respectively. In the first scenario, the lengths are unknown. This means that D_1 and D_2 are the unknown values. Rearrange the second equation in (1) so that:

$$D_2 = \frac{-\Delta y - D_1 \cos Az_1}{\cos Az_2}$$

This is then inserted into the first equation in (1) to determine the distance of the unknown leg, D_1

$$D_1 \sin Az_1 - \frac{\Delta y \sin Az_2 + D_1 \cos Az_1 \sin Az_2}{\cos Az_2} + \Delta x = 0$$

$$D_1 \sin Az_1 \cos Az_2 - \Delta y \sin Az_2 - D_1 \cos Az_1 \sin Az_2 + \Delta x \cos Az_2 = 0$$

$$D_1 = \frac{\Delta y \sin Az_2 - \Delta x \cos Az_2}{\sin Az_1 \cos Az_2 - \cos Az_1 \sin Az_2} \quad (2)$$

With D_1 known solve for D_2 using either (1a) or (1b). Stoughton [1975] simplifies this equation by recognizing the trig identity: $\sin(x - y) = \sin x \cos y - \cos x \sin y$. Therefore, (2) becomes

$$D_1 = \frac{\Delta y \sin Az_2 - \Delta x \cos Az_2}{\sin(Az_2 - Az_1)} \quad (3)$$

Stoughton presents this example: Given the following data:

Line	Azimuth	Distance
1	36^o 42’ 25”	468.38’
2	97^o 34’ 01”	D_2
3	193^o 02’ 56”	723.00’
4	222^o 15’ 08”	D_4
5	346^o 28’ 20”	967.30’

The sums of the eastings (Δx) and northings (Δy), or the departures and latitudes, for the known links are computed as

$$\Delta x = D_1 \sin Az_1 + D_3 \sin Az_3 + D_5 \sin Az_5$$

$$\Delta x = 468..38' \sin 36^o 42'25'' + 723.00' \sin 193^o 02'56'' + 967.30' \sin 246^o 28'20''$$

$$\Delta x = -109.547'$$

$$\Delta y = D_1 \cos Az_1 + D_3 \cos Az_3 + D_5 \cos Az_5$$

$$\Delta y = 468..38' \cos 36^o 42'25'' + 723.00' \cos 193^o 02'56'' + 967.30' \cos 246^o 28'20''$$

$$\Delta y = 611.635'$$

The difference in the azimuths is $Az_4 - Az_2 = 124^o$ 41’ 07”. Then, the distance, D_2 is computed using (3)

$$D_2 = \frac{\Delta y \sin Az_4 - \Delta x \cos Az_4}{\sin\left(Az_4 - Az_2\right)} = \frac{611.635' \sin 222^\circ 15'08'' - \left(-109.547'\right) \cos 222^\circ 15'08''}{\sin 124^\circ 41'07''}$$
$$= \underline{598.75'}$$

Alternatively, using (2), we can find the same value using the relationship

$$\begin{aligned} D_2 &= \frac{\Delta y \sin Az_4 - \Delta x \cos Az_4}{\sin Az_4 \cos Az_2 - \cos Az_4 \sin Az_2} \\ &= \frac{611.635 \sin 222^\circ 15'08'' - \left(-109.547\right) \cos 222^\circ 15'08''}{\sin\left(222^\circ 15'08''\right)\cos\left(97^\circ 34'01''\right) - \cos\left(222^\circ 15'08''\right)\sin\left(97^\circ 34'01''\right)} \\ &= 598.75' \end{aligned}$$

In the calculations, the distance is shown in a negative quantity. Since a distance cannot be negative, use the absolute value of this value. Finally, rearranging (1a) gives the distance to line 4.

$$D_4 = \frac{-D_2 \sin Az_2 - \Delta x}{\sin Az_4} = \frac{\left(-598.75 \sin 97^\circ 34'01''\right) - \left(-109.547\right)}{\sin 222^\circ 15'08''}$$
$$= \underline{719.80''}$$

The next category covers the problem where one length is unknown and the direction of another traverse course is unknown. Lets assume that Az_1 and D_2 are unknown. Designating M for the sin Az_2 and N for the cos Az_2, then (1a) and (1b) can be rewritten.

$$D_1 \sin Az_1 + D_2 M + \Delta x = 0$$

$$D_1 \cos Az_1 + D_2 N + \Delta y = 0$$

Rearrange into the following form:

$$\sin Az_1 = -\frac{D_2 M + \Delta x}{D_1} \qquad (4a)$$

$$\cos Az_1 = -\frac{D_2 N + \Delta y}{D_1} \qquad (4b)$$

Square each equation

$$\sin^2 Az_1 = \frac{(D_2 M + \Delta x)^2}{D_1^2} \tag{5a}$$

$$\cos^2 Az_1 = \frac{(D_2 N - \Delta y)^2}{D_1^2} \tag{5b}$$

Add (5a) and (5b)

$$\sin^2 Az_1 + \cos^2 Az_1 = \frac{(D_2 M + \Delta x)^2}{D_1^2} + \frac{(D_2 N + \Delta y)^2}{D_1^2}$$

$$1 = \frac{1}{D_1^2}\left[(D_2 M + \Delta x)^2 + (D_2 N + \Delta y)^2\right]$$

$$D_1^2 = D_2^2 M^2 + 2D_2 M\,\Delta x + \Delta x^2 + D_2^2 N^2 + 2D_2 N\,\Delta y + \Delta y^2$$

$$D_2^2\left(M^2 + N^2\right) + D_2\left(2M\Delta x + 2N\Delta y\right) + \left(\Delta x^2 + \Delta y^2 - D_1^2\right) = 0 \tag{6}$$

(6) is in the form of a quadratic equation and the solution is in the general forms as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

where:

$$a = M^2 + N^2 = \sin^2 Az_2 + \cos^2 Az_2$$
$$b = 2M\Delta x + 2N\Delta y = 2\Delta x \sin Az_2 + 2\Delta y \cos Az_2$$
$$c = \Delta x^2 + \Delta y^2 - D_1^2$$
$$x = D_2$$

With D_2 known, substitute into (4a) or (4b) and solve for Az_1. As we can see, there are two values for D_2. If any of the values are negative, these can be ignored. If both values are positive than an accurate sketch of the area is used to identify the proper values for D_2 and Az_1. Looking at $a = M^2 + N^2$, we see that $a = 1$. Also recognize that a 4 can be factored out of the radical. This leads to the solution of the quadratic equation shown in the following form (using the notation of Stoughton [1975])

$$D_1 = \frac{-2V \pm \sqrt{4V^2 - 4U}}{2} = -V \pm \sqrt{V^2 - U} \qquad (8)$$

where:

$U = \Delta x^2 + \Delta y^2 - D_2^2$, and

$V = \Delta x M + \Delta y N$

Example, from Stoughton [1975]. Given the following traverse data, compute D_2 and Az_4.

Line	Azimuth	Distance
1	36° 42’ 25”	468.38’
2	97° 34’ 01”	D_2
3	193° 02’ 56”	723.00’
4	Az_4	719.80
5	346° 28’ 20”	967.30’

The solution is as follows.

$$U = \Delta x^2 + \Delta y^2 - D_4^2 = (-109.547)^2 + (611.635)^2 - (719.80)^2 = -132{,}014.1216$$

$$V = \Delta x \sin Az_2 + \Delta y \cos Az_2 = (-109.547)\sin 97^\circ 34'01'' + (611.635)\cos 97^\circ 34'01'' = -189.13588$$

Substitute these values into (8) yields:

$$D_2 = -V \pm \sqrt{V^2 - U} = -(-189.13588) \pm \sqrt{(-189.13588)^2 - (-132{,}014.1216)}$$
$$= 598.75' \quad \text{or} \quad -220.48'$$

The last value is not possible therefore the solution is D_2 = 598.75’. Substitute this value into (4a) or (4b) to solve for the azimuth of line 4.

$$Az_4 = \sin^{-1}\left[\frac{D_2 M + \Delta x}{D_4}\right] = \sin^{-1}\left[\frac{-598.75 \sin 97^\circ 34'01'' - (-109.547)}{719.80}\right]$$
$$= -42^\circ\, 15'\, 08''$$

This result shows that the azimuth is in either the northwest or southwest quadrant. A good drawing of the traverse would show that it needs to be in the southwest quadrant therefore the azimuth is 222° 15’ 08”. This should be verified by computing the latitudes and departures for the traverse. For example, the azimuth of 222° 15’ 08” results in a closed traverse as shown in the accompanying spreadsheet.

Traverse Adjustment Program							
		Azimuth					
					Decimal		
Sta	Dist	Deg	Min	Sec	Degrees	Departure	Latitude
1							
	468.38	36	42	25	36.70694	279.96	375.50
2							
	598.75	97	34	1	97.56694	593.54	-78.85
3							
	723.00	193	2	56	193.04889	-163.24	-704.33
4							
	719.80	222	15	8.7	222.25242	-483.99	-532.79
5							
	967.30	346	28	20	346.47222	-226.27	940.46
1							
	3477.23					0.00	0.00

If the angle 317^o 44' 51.3" were used, the traverse data would be as shown in the next spreadsheet.

Traverse Adjustment Program							
		Azimuth					
					Decimal		
Sta	Dist	Deg	Min	Sec	Degrees	Departure	Latitude
1							
	468.38	36	42	25	36.70694	279.96	375.50
2							
	598.75	97	34	1	97.56694	593.54	-78.85
3							
	723.00	193	2	56	193.04889	-163.24	-704.33
4							
	719.80	317	44	51.3	317.74758	-483.99	532.79
5							
	967.30	346	28	20	346.47222	-226.27	940.46
1							
	3477.23					0.00	1065.58

Stoughton [1975] presents an alternative method of computing the azimuth of the missing line. His derivation is as follows. Multiply (1a) by -sin Az_1 and (1b) by cos Az_1.

$$-D_1 \sin Az_1 \cos Az_1 - D_2 \cos Az_2 \sin Az_1 - \Delta x \sin Az_1 = 0$$
$$D1 \sin Az_1 \cos Az_1 + D_2 \sin Az_2 \cos Az_1 + \Delta y \cos Az_1 = 0$$

Add the equations together and rearrange

$$D_2 \left(\sin Az_2 \cos Az_1 - \cos Az_2 \sin Az_1 \right) = \Delta x \sin Az_1 - \Delta y \cos Az_1$$

Divide this relationship by (D_2 cos Az_1) yields

$$\sin Az_2 - \tan Az_1 \cos Az_2 = \frac{\Delta x \sin Az_1 - \Delta y \cos Az_1}{D_2 \cos Az_1} \qquad (9)$$

Define: $\nu = \tan Az_1$ and

$$W = \frac{\Delta x \sin Az_1 - \Delta y \cos Az_1}{D_2 \cos Az_1}$$

Then, (9) becomes

$$\sin Az_2 - \nu \cos Az_2 = W$$

$$\sin Az_2 = W + \nu \cos Az_2 \qquad (10)$$

Recall the trig identity $\sin\alpha = \sqrt{1-\cos^2\alpha}$ which results in (10) taking on the form

$$\sqrt{1-\cos^2 Az_2} = W + \nu \cos Az_2$$

Square both sides and rearrange

$$1-\cos^2 Az_2 = W^2 + 2W\nu \cos Az_2 + \nu^2 \cos^2 Az_2$$

$$\left(\nu^2+1\right)\cos^2 Az_2 + 2W\nu \cos Az_2 + \left(W^2-1\right) = 0$$

In this form, use the quadratic equation to find the cos Az_2.

$$\cos Az_2 = \frac{-2W\nu \pm \sqrt{4W^2\nu^2 - 4\left(1+\nu^2\right)\left(W^2-1\right)}}{2\left(1+\nu^2\right)}$$

Factor out √4 yields the formula for computing the cosine of the azimuth.

$$\cos Az_2 = \frac{-W\nu \pm \sqrt{W^2\nu^2 - \left(W^2 + W^2\nu^2 - 1 - \nu^2\right)}}{\left(1+\nu 1\right)}$$

$$= \frac{-W\nu \pm \sqrt{\nu^2 + 1 - W^2}}{\left(1-\nu^2\right)}$$

Stoughton points out that there are four solution for the azimuth. Generally, two of the solutions can be rejected outright. To check the results of the remaining two, one should have a good-quality sketch from which the comparison can be made. Using both acceptable azimuths into equation (1) can help identify which one is correct for the particular problem.

The next category pertains to the situation where the direction of the two lines are unknown (i.e., Az_1 and Az_2 are unknown). Beginning with (1a) and (1b), rearrange into the following form:

$$\begin{aligned} D_1 \sin Az_1 &= -(D_2 \sin Az_2 + \Delta x) \\ D_1 \cos Az_1 &= -(D_2 \cos Az_2 + \Delta y) \end{aligned}$$

Square both sides

$$\begin{aligned} D_1^2 \sin^2 Az_1 &= (D_2 \sin Az_2 + \Delta x)^2 \\ D_1^2 \cos^2 Az_1 &= (D_2 \cos Az_2 + \Delta y)^2 \end{aligned}$$

Add these two equations yields

$$\begin{aligned} D_1^2 &= (D_2 \sin Az_2 + \Delta x)^2 + (D_2 \cos Az_2 + \Delta y)^2 \\ &= D_2^2 \sin^2 Az_2 + 2\,D_2(\sin Az_2)\Delta x + \Delta x^2 + D_2^2 \cos^2 Az_2 + 2\,D_2(\cos Az_2)\Delta y + \Delta y^2 \\ &= D_2^2 + 2\,D_2\left(\Delta x \sin Az_2 + \Delta y \cos Az_2 + \Delta x^2 + \Delta y^2\right) \end{aligned}$$

Rearranging

$$\Delta x \sin Az_2 + \Delta y \cos Az_2 = \frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2}$$

Divide both sides by $\sqrt{\Delta x^2 \Delta y^2}$

$$\frac{\Delta x \sin Az_2}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}} + \frac{\Delta y \cos Az_2}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}} = \frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)^{1/2}}$$

Let

$$\sin\theta = \frac{\Delta x}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}} \qquad ; \qquad \cos\theta = \frac{\Delta y}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}}$$

Then

$$\sin Az_2 \sin\theta + \cos Az_2 \cos\theta = \frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)^{1/2}} \qquad (11)$$

Using the trigonometric identity $\cos(u - v) = \sin u \sin v + \cos u \cos v$, (11) becomes

$$\cos(Az_2 - \theta) = \frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)^{1/2}}$$

$$Az_2 - \theta = \cos^{-1}\left[\frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)^{1/2}}\right] \qquad (12)$$

But, recall that

$$\theta = \cos^{-1}\left[\frac{\Delta y}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}}\right]$$

which when substituted back into (12) yields:

$$Az_2 = \cos^{-1}\left[\frac{D_1^2 - D_2^2 - \Delta x^3 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)}\right] + \cos^{-1}\left[\frac{\Delta y}{\left(\Delta x^2 + \Delta y^2\right)}\right] \qquad (13)$$

The main problem with this algorithm is that there are two possible solutions to the problem. The final determination of the azimuth of the second line will be dependent upon a good drawing of the area in questions. With this azimuth now known, the azimuth of the first line can be found using (1b) expressed in the following form.

$$Az_1 = \cos^{-1}\left[\frac{D_2 \cos Az_2 + \Delta y}{D_1}\right]$$

Stoughton [1975] presents a different form of this solution- that of a quadratic equation. Here, the azimuth of the second line is computed using (without derivation)

$$\cos Az_2 = \frac{-\Delta y\,J \pm \sqrt{\Delta x^2 + \Delta y^2 - \Delta x^2 J^2}}{\left(\dfrac{\Delta x^2 + \Delta y^2}{\Delta x}\right)}$$

where

$$J = \frac{\Delta x^2 + \Delta y^2 - D_1^2 + D_2^2}{2\,\Delta x D_2}$$

Stoughton then shows that the azimuth of the first line can be computed using the same formula except for the value J which is updated and referred to here as K. Thus,

$$\cos Az_1 = \frac{-\Delta y\,K \pm \sqrt{\Delta x^2 + \Delta y^2 - \Delta x^2 K^2}}{\left(\dfrac{\Delta x^2 + \Delta y^2}{\Delta x}\right)}$$

where

$$K = \frac{\Delta x^2 + \Delta y^2 - D_2^2 + D_1^2}{2\,\Delta x D_1}$$

Using the example presented in Stoughton [1975], the data are given as

Line	Azimuth	Distance
1	36° 42’ 25”	468.38’
2	Az_2	598.75’
3	193° 02’ 56”	723.00’
4	Az_4	719.80
5	346° 28’ 20”	967.30’

Then, using the formulas above, adjusted for the unknowns in this problem, compute J and the azimuth from 2-3.

$$J = \frac{\Delta x^2 + \Delta y^2 - D_2^2 + D_4^2}{2\,\Delta x\,D_4} = \frac{(-109.547)^2 + (611.635)^2 - (598.75)^2 + (719.80)^2}{2(-109.547)(719.80)}$$
$$= -3.460336$$

$$Az_4 = \cos^{-1}\left[\frac{-\Delta y\,J \pm \sqrt{\Delta x^2 + \Delta y^2 - \Delta x^2 J^2}}{\left(\dfrac{\Delta x^2 + \Delta y^2}{\Delta x}\right)}\right]$$

$$= \cos^{-1}\left[\frac{-611.635(-3.460336) \pm \sqrt{(-109.547)^2 + (611.635)^2 - (-109.547)^2(-3.460336)^2}}{\left(\dfrac{(-109.547)^2 + (611.635)^2}{-109.547}\right)}\right]$$

$$= \cos^{-1}\left[\frac{2{,}116.46279 \pm \sqrt{242{,}404.2622}}{-3{,}524.4956}\right]$$

As Stoughton points out, there are four possible answers to this quadratic equation. Recall that with the ambiguous case that the angle could be either the value determined directly from the computation or its complement. The possible solutions are 137° 44' 52.4" or 222° 15' 07.6" when the radical is added and 117° 26' 21.4" or 242° 33' 38.6" when the radical is subtracted. Then, solve for the azimuth between points 2 and 3.

$$K = \frac{\Delta x^2 + \Delta y^2 - D_4^2 + D_2^2}{2\,\Delta x\,D_2}$$
$$= -1.726506$$

$$Az_2 = \cos^{-1}\left[\frac{-\Delta y\,K \pm \sqrt{\Delta x^2 + \Delta y^2 - \Delta x^2 K^2}}{\left(\dfrac{\Delta x^2 + \Delta y^2}{\Delta x}\right)}\right] = \cos^{-1}\left[\frac{1{,}055.9916 \pm \sqrt{350{,}326.4140}}{-3{,}524.4956}\right]$$

As before, there are four solutions to this quadratic equation. They are: 117°52' 31.2" or 242° 07' 28.8" when the radical is added and 97° 34' 00.2" or 262° 25' 59.8" when the radical is subtracted

A good drawing of the traverse would show what two angles fit the situation the best. Absent a drawing, there will be two solutions. These are shown in the following two spreadsheets.

Traverse calculation using the solution set of Az_2 = 97° 34' 00.2" and Az_1 = 222° 15' 07.6"is shown as:

Traverse Calculation Program							
		Azimuth					
					Decimal		
Sta	Dist	Deg	Min	Sec	Degrees	Departure	Latitude
1							
	468.38	36	42	25	36.70694	279.961	375.502
2							
	598.75	97	34	0.2	97.56672	593.536	-78.844
3							
	723.00	193	2	56	193.04889	-163.241	-704.331
4							
	719.80	222	15	7.6	222.25211	-483.989	-532.791
5							
	967.30	346	28	20	346.47222	-226.268	940.464
1							
	3477.23					0.000	0.000

Traverse calculation using the solution set of Az_2 = 242° 07' 28.8" and Az_1 = 117° 26' 21.4"is shown as:

Traverse Calculation Program							
		Azimuth					
					Decimal		
Sta	Dist	Deg	Min	Sec	Degrees	Departure	Latitude
1							
	468.38	36	42	25	36.70694	279.961	375.502
2							
	598.75	242	7	28.8	242.12467	-529.275	-279.945
3							
	723.00	193	2	56	193.04889	-163.241	-704.331
4							
	719.80	117	26	21.4	117.43928	638.822	-331.690
5							
	967.30	346	28	20	346.47222	-226.268	940.464
1							
	3477.23					0.000	0.000

The other solution possibilities do not yield acceptable results when inserted in the traverse calculations program.

CONCLUSION

As it has been shown, computing missing data in a polygon requires different solutions, depending on what is unknown. The solution can be found using a geometric reconstruction of the problem or by analytical methods. The former may be more intuitive but does require more computations.

The problem may have no unique solution. This occurs when the directions of two sides are unknown. In such a situation, a good sketch of the field conditions will be required to resolve this ambiguity.

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