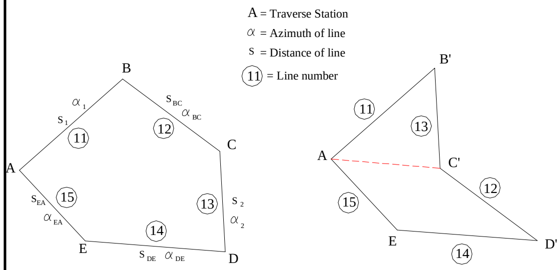
COMPUTING MISSING DATA IN A POLYGON

Robert Burtch
Surveying Engineering
Ferris State University

TRAVERSE WITH MISSING ELEMENTS

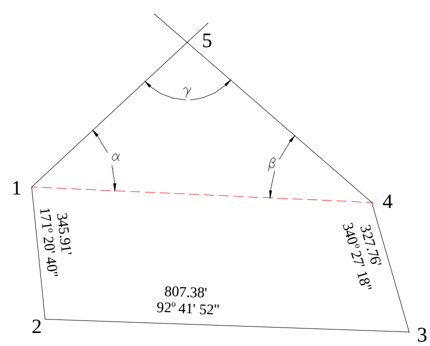


TRAVERSE WITH MISSING ELEMENTS

3 forms of missing data problem

1) Missing distance and direction of same line – trivial problem
2) Missing 2 distances of 2 adjacent lines, or 2 directions of 2 adjacent lines, or distance and direction of 2 adjacent lines
3) 2 unknown distances or directions on 2 non-adjacent lines or unknown distance on one line and unknown direction on another non-adjacent line

MISSING DATA ON 2 ADJACENT LINES



EXAMPLE ---MISSING DATA ON 2 ADJACENT LINES

STA	DISTANCE	AZIMUTH	X	Y	COMMENTS
1			1000.00	500.00	Assumed
	345.91	171° 20' 40"			
2			1052.06	158.03	
	807.38	92° 41' 52"			
3			1858.54	120.03	
	327.76	340° 27' 18"			
4			1748.89	428.90	

- Compute distance and azimuth of line 4-1

$$D_{4-1} = \sqrt{(X_1 - X_4)^2 + (Y_1 - Y_4)^2} = \sqrt{(1000.00 - 1748.89)^2 + (500.00 - 428.90)^2}$$

$$D_{4-1} = 752.26'$$

$$Az_{4-1} = \tan^{-1}\left[\frac{X_1 - X_4}{Y_1 - Y_4}\right] = \tan^{-1}\left[\frac{(1000.00 - 1748.89)}{(500.00 - 428.90)}\right]$$

$$Az_{4-1} = 275^\circ\ 25'\ 24''$$

- Solve of missing elements in triangle 1-4-5

Solve for interior angles

$$\alpha = Az_{1-4} - Az_{1-5} = (95^\circ 25' 24'') - (33^\circ 47' 32'')$$

$$\alpha = 61^\circ\ 37'\ 52''$$

$$\beta = Az_{4-5} - Az_{4-1} = (314^\circ 51' 18'') - (275^\circ 25' 24'')$$

$$\beta = 39^\circ\ 25'\ 54''$$

$$\gamma = 180^\circ - (\alpha + \beta) = 180^\circ - \left[(61^\circ\ 37'\ 52'') + (39^\circ\ 25'\ 54'')\right]$$

$$\gamma = 78^\circ\ 56'\ 14''$$

- Using sine law, distances can be computed

$$\frac{D_{4-5}}{\sin\alpha} = \frac{D_{1-5}}{\sin\beta} = \frac{D_{1-4}}{\sin\gamma}$$

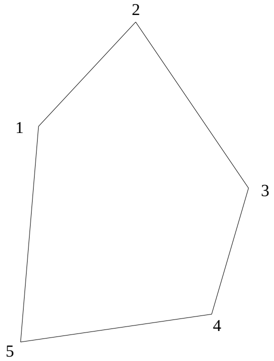
$$D_{1-5} = \left(\frac{D_{1-4}}{\sin\gamma}\right)\sin\beta = \left(\frac{752.26}{\sin 78^\circ 56' 14''}\right)\sin 39^\circ 25' 54''$$

$$D_{1-5} = 486.85'$$

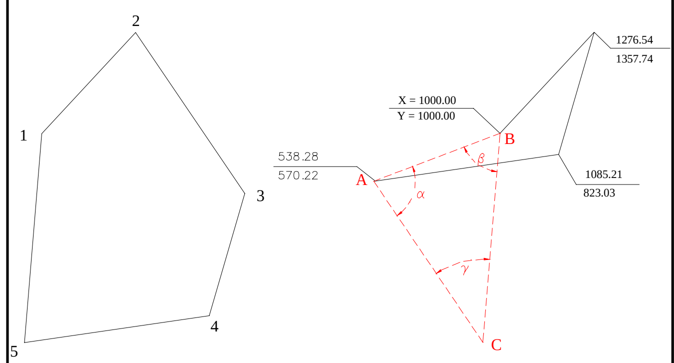
$$D_{4-5} = \left(\frac{D_{1-4}}{\sin\gamma}\right)\sin\alpha = \left(\frac{752.26}{\sin 78^\circ 56' 14''}\right)\sin 61^\circ 37' 52''$$

$$D_{4-5} = 674.45'$$

MISSING DATA ON 2 NON-ADJACENT LINES



STA	Distance	Azimuth
1		
	452.97'	37°42'17"
2		
	?	145°14'20"
3		
	567.92'	199°41'19"
4		
	602.53'	245°11'32"
5		
	?	2°21'43"
6		



EXAMPLE

- Compute distance and direction between A and B

$$D_{AB} = \sqrt{(X_B - X_A)^2 + (Y_B - Y_C)^2}$$
$$= \sqrt{(1000.00 - 538.28)^2 + (1000.00 - 570.22)^2}$$
$$D_{AB} = 630.79'$$

$$Az_{AB} = \tan^{-1}\left[\frac{(X_B - X_A)}{(Y_B - Y_C)}\right] = \tan^{-1}\left[\frac{(1000.00 - 538.28)}{(1000.00 - 570.22)}\right]$$
$$Az_{AB} = 47^\circ 03'07''$$

angles α β, and γ can be computed.

$$\alpha = Az_{AC} - Az_{AB}$$
$$= (145^\circ 14'20'') - (47^\circ 03'07'') = 98^\circ 11'13''$$

$$\beta = Az_{BA} - Az_{BC}$$
$$= (227^\circ 03'07'') - (182^\circ 21'43'') = 44^\circ 41'24''$$

$$\gamma = 360^\circ - (Az_{CA} - Az_{CB})$$
$$= 360^\circ - [(325^\circ 14'20'') - (2^\circ 21'43'')]$$
$$= 37^\circ 07'23''$$

Use sine law to compute distances

$$D_{BC} = \left(\frac{D_{AB}}{\sin\gamma}\right)\sin\alpha$$
$$= \left(\frac{630.79}{\sin 37^o 07'23"}\right)\sin 98^o 11'13"$$
$$= \underline{1034.52}$$

$$D_{AC} = \left(\frac{D_{AB}}{\sin\gamma}\right)\sin\beta$$
$$= \left(\frac{630.79}{\sin 37^o 07'23"}\right)\sin 44^o 41'24"$$
$$= \underline{735.04}$$

Check results

STA	DISTANCE	AZIMUTH	LATITUDE	DEPARTURE
1				
	452.17	37° 42' 17"	+357.74	+276.54
2				
	735.04	145° 14' 20"	-603.86	+419.09
3				
	567.92	199° 41' 19"	-534.72	-191.34
4				
	602.53	245° 11' 32"	-252.80	-546.92
5				
	1034.52	2° 21' 43"	+1033.64	+42.63
1				
			0.00	0.00

ANALYTICAL APPROACH

- Exploit fact that in closed loop traverse that the sum of latitudes and departures is zero
- Write 2 equations with 2 unknowns
- Equations solved simultaneously
- Unknowns in 3 categories
  - 2 unknowns both lengths of traverse
  - 1 length unknown and 1 direction of another traverse course
  - Direction of 2 lines unknowns

CATEGORY 1

- Write equations

$$D_1 \sin Az_1 + D_2 \sin Az_2 + \Delta x = 0$$

$$D_1 \cos Az_1 + D_2 \cos Az_2 + \Delta y = 0$$

- Rearrange 2nd equation

$$D_2 = \frac{-\Delta y - D_1 \cos Az_1}{\cos Az_2}$$

CATEGORY 1

$$D_1 \sin Az_1 - \frac{\Delta y \sin Az_2 + D_1 \cos Az_1 \sin Az_2}{\cos Az_2} + \Delta x = 0$$

$$D_1 \sin Az_1 \cos Az_2 - \Delta y \sin Az_2 - D_1 \cos Az_1 \sin Az_2 + \Delta x \cos Az_2 = 0$$

$$D_1 = \frac{\Delta y \sin Az_2 - \Delta x \cos Az_2}{\sin Az_1 \cos Az_2 - \cos Az_1 \sin Az_2}$$

Recognizing: $\sin(x - y) = \sin x \cos y - \cos x \sin y$

$$D_1 = \frac{\Delta y \sin Az_2 - \Delta x \cos Az_2}{\sin(Az_2 - Az_1)}$$

CATEGORY 1 EXAMPLE

Given data:

Line	Azimuth	Distance
1	36^o 42’ 25”	468.38’
2	97^o 34’ 01”	D_2
3	193^o 02’ 56”	723.00’
4	222^o 15’ 08”	D_4
5	346^o 28’ 20”	967.30’

CATEGORY 1 EXAMPLE

$$\Delta x = D_1 \sin Az_1 + D_3 \sin Az_3 + D_5 \sin Az_5$$

$$\Delta x = 468..38' \sin 36^o 42'25'' + 723.00' \sin 193^o 02'56'' + 967.30' \sin 246^o 28'20''$$

$$\Delta x = -109.547'$$

$$\Delta y = D_1 \cos Az_1 + D_3 \cos Az_3 + D_5 \cos Az_5$$

$$\Delta y = 468..38' \cos 36^o 42'25'' + 723.00' \cos 193^o 02'56'' + 967.30' \cos 246^o 28'20''$$

$$\Delta y = 611.635'$$

Difference in azimuths from 2 to 4:

Az_4 - Az_2 = 124º 41’ 07”

CATEGORY 1 EXAMPLE

- The distance D_2 is

$$D_2 = \frac{\Delta y \sin Az_4 - \Delta x \cos Az_4}{\sin(Az_4 - Az_2)} = \frac{611.635' \sin 222^o 15'08'' - (-109.547') \cos 222^o 15'08''}{\sin 124^o 41'07''}$$

$$= \underline{598.75'}$$

- Alternatively, can also compute D_2 as

$$D_2 = \frac{\Delta y \sin Az_4 - \Delta x \cos Az_4}{\sin Az_4 \cos Az_2 - \cos Az_4 \sin Az_2}$$

$$= \frac{611.635 \sin 222^o 15'08'' - (-109.547) \cos 222^o 15'08''}{\sin(222^o 15'08'') \cos(97^o 34'01'') - \cos(222^o 15'08'') \sin(97^o 34'01'')}$$

$$= 598.75'$$

CATEGORY 1 EXAMPLE

Distance D_4 found from

$$D_4 = \frac{-D_2 \sin Az_2 - \Delta x}{\sin Az_4} = \frac{(-598.75 \sin 97^\circ 34'01'') - (-109.547)}{\sin 222^\circ 15'08''}$$
$$= \underline{719.80''}$$

CATEGORY 2

- One length unknown and direction of another course unknown
- Designate **M** for sin Az_2 and **N** for cos Az_2:

$$D_1 \sin Az_1 + D_2 M + \Delta x = 0$$

$$D_1 \cos Az_1 + D_2 N + \Delta y = 0$$

- Rearrange

$$\sin Az_1 = -\frac{D_2 M + \Delta x}{D_1}$$

$$\cos Az_1 = -\frac{D_2 N + \Delta y}{D_1}$$

CATEGORY 2

$$\sin^2 Az_1 + \cos^2 Az_1 = \frac{(D_2 M + \Delta x)^2}{D_1^2} + \frac{(D_2 N + \Delta y)^2}{D_1^2}$$

$$1 = \frac{1}{D_1^2}\left[(D_2 M + \Delta x)^2 + (D_2 N + \Delta y)^2\right]$$

$$D_1^2 = D_2^2 M^2 + 2D_2 M \Delta x + \Delta x^2 + D_2^2 N^2 + 2D_2 N \Delta y + \Delta y^2$$

$$D_2^2(M^2 + N^2) + D_2(2M\Delta x + 2N\Delta y) + (\Delta x^2 + \Delta y^2 - D_1^2) = 0$$

CATEGORY 2

Form of quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where

$$a = M^2 + N^2 = \sin^2 Az_2 + \cos^2 Az_2$$
$$b = 2M\Delta x + 2N\Delta y = 2\Delta x \sin Az_2 + 2\Delta y \cos Az_2$$
$$c = \Delta x^2 + \Delta y^2 - D_1^2$$
$$x = D_2$$

CATEGORY 2

- 4 can be factored out of quadratic equation leading to this solution

$$D_1 = \frac{-2V \pm \sqrt{4V^2 - 4U}}{2}$$
$$= -V \pm \sqrt{V^2 - U}$$

where:

$$U = \Delta x^2 + \Delta y^2 - D_2^2, \quad \text{and}$$
$$V = \Delta x M + \Delta y N$$

CATEGORY 2 EXAMPLE

Line	Azimuth	Distance
1	36^{o} 42’ 25”	468.38’
2	97^{o} 34’ 01”	D_2
3	193^{o} 02’ 56”	723.00’
4	Az_4	719.80
5	346^{o} 28’ 20”	967.30’

CATEGORY 2 EXAMPLE

- Solution

$$U = \Delta x^2 + \Delta y^2 - D_4^2$$
$$= (-109.547)^2 + (611.635)^2 - (719.80)^2 = -132{,}014.1216$$
$$V = \Delta x \sin Az_2 + \Delta y \cos Az_2$$
$$= (-109.547)\sin 97^{o}34'01'' + (611.635)\cos 97^{o}34'01'' = -189.13588$$

- Distance:

$$D_2 = -V \pm \sqrt{V^2 - U}$$
$$= -(-189.13588) \pm \sqrt{(-189.13588)^2 - (-132{,}014.1216)}$$
$$= 598.75' \text{or} - 220.48'$$

- Ignore second distance

CATEGORY 2 EXAMPLE

- Solve for azimuth of line 4

$$Az_4 = \sin^{-1}\left[\frac{D_2 M + \Delta x}{D_4}\right] = \sin^{-1}\left[\frac{-598.75 \sin 97^{o}34'01'' - (-109.547)}{719.80}\right]$$
$$= -42^{o}\,15'08''$$

- NW or SW quadrant?
  - Good drawing: 222^{o} 15’ 08”

CATEGORY 2 EXAMPLE SW QUADRANT RESULT

Traverse Adjustment Program							
		Azimuth					
					Decimal		
Sta	Dist	Deg	Min	Sec	Degrees	Departure	Latitude
1							
	468.38	36	42	25	36.70694	279.96	375.50
2							
	598.75	97	34	1	97.56694	593.54	-78.85
3							
	723.00	193	2	56	193.04889	-163.24	-704.33
4							
	719.80	222	15	8.7	222.25242	-483.99	-532.79
5							
	967.30	346	28	20	346.47222	-226.27	940.46
1							
	3477.23					0.00	0.00

CATEGORY 2 EXAMPLE NW QUADRANT RESULT

Traverse Adjustment Program							
		Azimuth					
					Decimal		
Sta	Dist	Deg	Min	Sec	Degrees	Departure	Latitude
1							
	468.38	36	42	25	36.70694	279.96	375.50
2							
	598.75	97	34	1	97.56694	593.54	-78.85
3							
	723.00	193	2	56	193.04889	-163.24	-704.33
4							
	719.80	317	44	51.3	317.74758	-483.99	532.79
5							
	967.30	346	28	20	346.47222	-226.27	940.46
1							
	3477.23					0.00	1065.58

CATEGORY 2 ALTERNATIVE

- Multiply 1st and 2nd equation in slide 16 by -sin Az_1 and by cos Az_1 respectively

$$-D_1 \sin Az_1 \cos Az_1 - D_2 \cos Az_2 \sin Az_1 - \Delta x \sin Az_1 = 0$$
$$D1 \sin Az_1 \cos Az_1 + D_2 \sin Az_2 \cos Az_1 + \Delta y \cos Az_1 = 0$$

- After manipulation

$$\cos Az_2 = \frac{-W\nu \pm \sqrt{W^2\nu^2 - \left(W^2 + W^2\nu^2 - 1 - \nu^2\right)}}{(1+\nu 1)}$$
$$= \frac{-W\nu \pm \sqrt{\nu^2 + 1 - W^2}}{\left(1-\nu^2\right)}$$

CATEGORY 3

- Square both equation on slide 16

$$D_1^2 \sin^2 Az_1 = \left(D_2 \sin Az_2 + \Delta x\right)^2$$
$$D_1^2 \cos^2 Az_1 = \left(D_2 \cos Az_2 + \Delta y\right)^2$$

- Add equations

$$D_1^2 = \left(D_2 \sin Az_2 + \Delta x\right)^2 + \left(D_2 \cos Az_2 + \Delta y\right)^2$$
$$= D_2^2 \sin^2 Az_2 + 2 D_2 \left(\sin Az_2\right)\Delta x + \Delta x^2 + D_2^2 \cos^2 Az_2 + 2 D_2 \left(\cos Az_2\right)\Delta y + \Delta y^2$$
$$= D_2^2 + 2 D_2 \left(\Delta x \sin Az_2 + \Delta y \cos Az_2 + \Delta x^2 + \Delta y^2\right)$$

CATEGORY 3

- Rearrange

$$\Delta x \sin Az_2 + \Delta y \cos Az_2 = \frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2}$$

- Divide by $\sqrt{\Delta x^2 \Delta y^2}$

$$\frac{\Delta x \sin Az_2}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}} + \frac{\Delta y \cos Az_2}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}} = \frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)^{1/2}}$$

CATEGORY 3

- Let $\sin\theta = \frac{\Delta x}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}};\cos\theta = \frac{\Delta y}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}}$
- And using trig identity
  $\cos(u - v) = \sin u \sin v + \cos u \cos v$
- then

$$\cos(Az_2 - \theta) = \frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)^{1/2}}$$

$$Az_2 - \theta = \cos^{-1}\left[\frac{D_1^2 - D_2^2 - \Delta x^2 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)^{1/2}}\right]$$

CATEGORY 3

- Recall, $\theta = \cos^{-1}\left[\frac{\Delta y}{\left(\Delta x^2 + \Delta y^2\right)^{1/2}}\right]$
- yields

$$Az_2 = \cos^{-1}\left[\frac{D_1^2 - D_2^2 - \Delta x^3 - \Delta y^2}{2\,D_2\left(\Delta x^2 + \Delta y^2\right)}\right] + \cos^{-1}\left[\frac{\Delta y}{\left(\Delta x^2 + \Delta y^2\right)}\right]$$

$$Az_1 = \cos^{-1}\left[\frac{D_2 \cos Az_2 + \Delta y}{D_1}\right]$$

CATEGORY 3 ALTERNATIVE

$$\cos Az_2 = \frac{-\Delta y\,J \pm \sqrt{\Delta x^2 + \Delta y^2 - \Delta x^2 J^2}}{\left(\frac{\Delta x^2 + \Delta y^2}{\Delta x}\right)}$$

where:

$$J = \frac{\Delta x^2 + \Delta y^2 - D_1^2 + D_2^2}{2\,\Delta x D_2}$$

CATEGORY 3 ALTERNATIVE

$$\cos Az_1 = \frac{-\Delta y\, K \pm \sqrt{\Delta x^2 + \Delta y^2 - \Delta x^2 K^2}}{\left(\dfrac{\Delta x^2 + \Delta y^2}{\Delta x}\right)}$$

where

$$K = \frac{\Delta x^2 + \Delta y^2 - D_2^2 + D_1^2}{2\,\Delta x D_1}$$

CATEGORY 3 EXAMPLE

Line	Azimuth	Distance
1	36^o 42’ 25”	468.38’
2	Az_2	598.75’
3	193^o 02’ 56”	723.00’
4	Az_4	719.80
5	346^o 28’ 20”	967.30’

CATEGORY 3 EXAMPLE

$$J = \frac{\Delta x^2 + \Delta y^2 - D_2^2 + D_4^2}{2\,\Delta x\, D_4} = \frac{(-109.547)^2 + (611.635)^2 - (598.75)^2 + (719.80)^2}{2(-109.547)(719.80)}$$

$$= -3.460336$$

$$Az_4 = \cos^{-1}\left[\frac{-\Delta y J \pm \sqrt{\Delta x^2 + \Delta y^2 - \Delta x^2 J^2}}{\left(\dfrac{\Delta x^2 + \Delta y^2}{\Delta x}\right)}\right]$$

$$= \cos^{-1}\left[\frac{2{,}116.46279 \pm \sqrt{242{,}404.2622}}{-3{,}524.4956}\right]$$

Solution set of Az_2 = 97° 34’ 00.2” and Az_1 = 222° 15’ 07.6”

Traverse Calculation Program							
		Azimuth					
					Decimal		
Sta	Dist	Deg	Min	Sec	Degrees	Departure	Latitude
1							
	468.38	36	42	25	36.70694	279.961	375.502
2							
	598.75	97	34	0.2	97.56672	593.536	-78.844
3							
	723.00	193	2	56	193.04889	-163.241	-704.331
4							
	719.80	222	15	7.6	222.25211	-483.989	-532.791
5							
	967.30	346	28	20	346.47222	-226.268	940.464
1							
	3477.23					0.000	0.000

Solution set of Az_2 = 242° 07' 28.8" and Az_1 = 117° 26' 21.4"

Traverse Calculation Program							
		Azimuth					
					Decimal		
Sta	Dist	Deg	Min	Sec	Degrees	Departure	Latitude
1							
	468.38	36	42	25	36.70694	279.961	375.502
2							
	598.75	242	7	28.8	242.12467	-529.275	-279.945
3							
	723.00	193	2	56	193.04889	-163.241	-704.331
4							
	719.80	117	26	21.4	117.43928	638.822	-331.690
5							
	967.30	346	28	20	346.47222	-226.268	940.464
1							
	3477.23					0.000	0.000