LINE-LINE INTERSECTION

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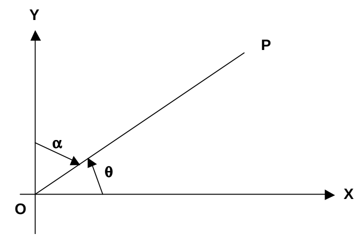
EQUATION OF A LINE

- Slope



$$m = \tan\theta = \cot\alpha = \frac{\text{rise}}{\text{run}}$$

EQUATION OF A LINE

- Ratios of difference in coordinates between 2 points on straight line are equal
- General equation: $$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
- Rearrange:

$$aX + bY + c = 0$$

  – where: $a = y_1 - y_2$
    $b = x_2 - x_1$, and
    $c = x_1 (y_2 - y_1) - y_1 (x_2 - x_1)$

EQUATION OF A LINE - EXAMPLE

- Write equation of line passing through A(4,-3) and B(10,5)
- Solution: $a = y_A - y_B = -3 - 5 = -8$
  $b = x_B - x_A = 10 - 4 = 6$
  $c = x_A(y_B - y_A) - y_A(x_B - x_A)$
  $= 4(5 - (-3)) - (-3)(10 - 4) = 50$
- Equation of line:

$$\underline{-4x + 3y + 25 = 0}$$

EQUATION OF A LINE - EXAMPLE

- Alternative solution:

$$\frac{y+3}{5+3} = \frac{x-4}{10-4} \quad \Rightarrow \quad \frac{y+3}{8} = \frac{x-4}{6}$$

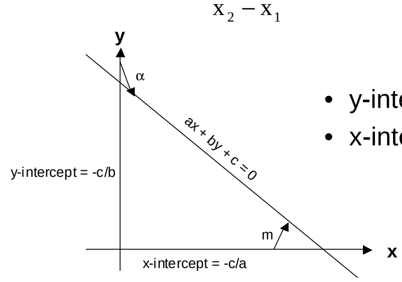
$$6y + 18 = 8x - 32 \quad \Rightarrow \quad -8x + 6y + 50 = 0$$

$$\underline{-4x + 3y + 25 = 0}$$

GEOMETRY OF A STRAIGHT LINE

- Slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad m = -\frac{a}{b}$$



- y-intercept: -c/b
- x-intercept: -c/a

GEOMETRY OF A STRAIGHT LINE

- Azimuth

$$\alpha = \tan^{-1}\left[\frac{x_2 - x_1}{y_2 - y_1}\right] = \tan^{-1}\left(-\frac{b}{a}\right)$$

- From example

$$m = -\frac{(-8)}{6} = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left[-\frac{6}{(-8)}\right] = \tan^{-1}(0.75) = 36^\circ 52' 12''$$

GEOMETRY OF A STRAIGHT LINE

- Expand relationship by looking at slope and coordinates of one point

$$y - y_1 = \cot\alpha\,(x - x_1)$$

- Rearrange $-\cot\alpha * x + y - y_1 + \cot\alpha * x_1 = 0$
- Then $ax + by + c = 0$
  - where $a = -\cot\alpha = -m$
    $b = 1$
    $c = x_1 \cot\alpha - y_1 = mx_1 - y_1$

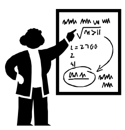
GEOMETRY OF A STRAIGHT LINE

- From example

$$a = -4/3$$
$$b = 1$$

- Equation of line: $c = \left(\frac{4}{3}\right)4 - (-3) = \frac{16}{3} + \frac{9}{3} = \frac{25}{3}$

$$\frac{-4x}{3} + y + \frac{25}{3} = 0$$



$$-4x + 3y + 25 = 0$$

SOLVING LINEAR EQUATIONS

- Equation for 2 lines

$$4x + 5y + 12 = 0$$
$$12x + 30y + 20 = 0$$

- First, multiply first equation by 3 and add it to the second

$$-12x - 15y - 36 = 0$$
$$+12x + 30y + 20 = 0$$
$$15y - 16 = 0$$

$$y = 16/15$$

SOLVING LINEAR EQUATIONS

- Solve for x

$$4x + 5\left(16/15\right) + 12 = 0$$
$$4x + 16/3 + 12 = 0$$
$$12x = -52$$
$$x = -\frac{52}{12} = -\frac{13}{3}$$

- Check results

$$12x + 30y + 20 = 0$$
$$12\left(-13/3\right) + 30\left(16/15\right) + 20 = 0$$
$$-52 + 32 + 20 = 0$$
$$0 = 0 \quad \text{(check)}$$

SOLVING LINEAR EQUATIONS

- Alternatively, isolate one variable, like x in first equation

$$4x + 5y + 12 = 0$$
$$4x = -5y - 12$$
$$x = -5/4\, y - 3$$

- Substitute into the second equation

$$12\left(-5/4\, y - 3\right) + 30y + 20 = 0$$
$$-15y - 36 + 30y + 20 = 0$$
$$15y = 16 \quad \Rightarrow \quad y = 16/15$$

LINE-LINE INTERSECTION

- 3 Methods
  - Simultaneous solution of the equations of 2 intersecting lines
  - Using the triangle solution
  - Base angle method

SIMULTANEOUS SOLUTION

- In first equation,isolat e variable like y

$$a_1x + b_1y + c_1 = 0$$

$$y = \frac{-(a_1x + c_1)}{b_1}$$

- Substitute y into 2nd equation, solve for x

$$a_2x + b_2y + c_2 = 0$$

$$a_2x + b_2\left[\frac{-(a_1x + c_1)}{b_1}\right] + c_2 = 0$$

$$a_2b_1x - a_1b_2x - b_2c_1 + b_1c_2 = 0$$

$$(a_2b_1 - a_1b_2)x = b_2c_1 - b_1c_2$$

$$x = \frac{b_2c_1 - b_1c_2}{a_2b_1 - a_1b_2}$$

SIMULTANEOUS SOLUTION

- If slope is given: $y - y_1 = m_1(x - x_1)$

$$y - y_2 = m_2(x - x_2)$$

- Then: $a = -m$

$$b = 1$$

$$c = mx - y$$

SIMULTANEOUS SOLUTION

- Solve for x and y

$$x = \frac{m_1x_1 - y_1 - (m_2x_2 - y_2)}{-m_2 - (-m_1)}$$

$$= \frac{m_1x_1 - m_2x_2 - y_1 + y_2}{m_1 - m_2}$$

$$y = \frac{-(-m_1x + m_1x_1 - y_1)}{1}$$

$$= m_1(x - x_1) + y_1$$

EXAMPLE

- Given:

$x_1 = 1254.52$	$x_2 = 1477.34$
$y_1 = 2001.94$	$y_2 = 1788.72$
$m_1 = 0.9103909$	$m_2 = -6.4797612$

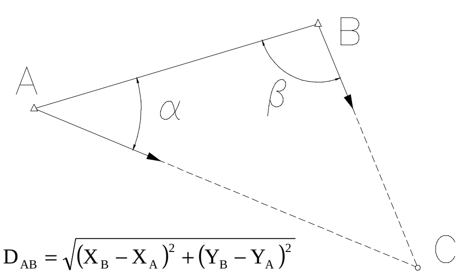
- Find the point of intersection

SOLUTION TO EXAMPLE

$$x = \frac{m_1 x_1 - m_2 x_2 - y_1 + y_2}{m_1 - m_2}$$
$$= \frac{(0.9103909)(1254.52) - (-6.4797612)(1477.34) - 2001.94 + 1788.72}{0.9103909 - (-6.4797612)}$$
$$= \underline{1421.04}$$

$$y = m_1(x - x_1) + y_1$$
$$= (0.9103909)(1421.03 - 1254.52) + 2001.94$$
$$= \underline{2153.54}$$

TRIANGLE SOLUTION



$$D_{AB} = \sqrt{(X_B - X_A)^2 + (Y_B - Y_A)^2}$$

$$Az_{AB} = \tan^{-1}\left[\frac{X_B - X_A}{Y_B - Y_A}\right]$$

TRIANGLE SOLUTION

- Solve for missing elements of triangle

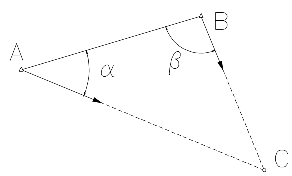
$$\angle_C = 180^\circ - (\alpha + \beta)$$

$$\frac{\sin\alpha}{D_{BC}} = \frac{\sin\beta}{D_{AC}} = \frac{\sin\angle_C}{D_{AB}} \Rightarrow \quad D_{AC} = \sin\beta\left(\frac{D_{AB}}{\sin\angle_C}\right)$$

$$D_{BC} = \sin\alpha\left(\frac{D_{AB}}{\sin\angle_C}\right)$$

BASE ANGLE METHOD

- Direct solution by solving for only the known values
- Method developed by expanding equations used in triangle method



BASE ANGLE METHOD

- x-coordinate found by

$$X_C = X_A + \frac{\sin\beta}{\sin(\alpha+\beta)}\left[(X_B - X_A)\cos\alpha + (Y_B - Y_A)\sin\alpha\right]$$

- or,

$$X_C = \frac{(Y_B - Y_A) + X_A \cot\beta + X_B \cot\alpha}{\cot\alpha + \cot\beta}$$

BASE ANGLE METHOD

- y-coordinate found by

$$Y_C = Y_A + \frac{\sin\beta}{\sin(\alpha+\beta)}\left[(Y_B - Y_A)\cos\alpha - (X_B - X_A)\sin\alpha\right]$$

- or,

$$Y_C = \frac{(X_A - X_B) + Y_A \cot\beta + Y_B \cot\alpha}{\cot\alpha + \cot\beta}$$

BASE ANGLE METHOD

- It is very important to look at the sign of the angles α and β in applying the base angle method. Standing at point A and looking towards point B, if the azimuths put the intersection to the right then α and β are positive. If they are to the left then α and β are negative.