4 LINE-CIRCLE INTERSECTION

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INTRODUCTION

There are three relationships within coordinate geometry that can exist between a line and a circle. They can intersect at one or two points or they will not intersect. If there is only one intersection then the line must be tangent to the circle at that point of intersection. To find where the intersection exists, one can use either the solution of two equations or by the triangle method.

To solve the problem in a simple mode, one can solve for the equation of a line and the equation of the circle simultaneously. Since the equation of a circle is a second-degree equation, the solution is through the quadratic equation. The development of the solution by simultaneous equations begins by writing the equation of the line and circle in the following forms (Hashimi, 1988):

$$y - y_1 = m(x - x_1) \rightarrow y = m(x - x_1) + y_1$$

$$R^2 = (x - h)^2 + (y - k)^2$$

where: x and y are the coordinates of the point of intersection
x_1 and y_1 are the coordinates of a point on the line
h and k are the translation between the center of the circle and the origin of the coordinate system in the x and y directions respectively.
m is the slope of the line.

As a reminder, the slope is the cotangent of the azimuth of the line. Write the equation of the line and expand the equation by taking the squares and equating the equation to zero.

$$(x - h)^2 + (y - k)^2 - R^2 = 0$$

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 - R^2 = 0 \qquad (1)$$

Substitute into this equation the relationship developed for the y-coordinate above, yielding:

$$x^2 - 2hx + h^2 + [m(x - x_1) + y_1]^2 - 2k[m(x - x_1) + y_1] + k^2 - R^2 = 0$$

$$x^2 - 2hx + h^2 + m^2x^2 - 2m^2xx_1 + 2mxy_1 + m^2x_1^2 - 2mx_1y_1 + y_1{}^2 \qquad (2)$$

$$-2kmx + 2kmx_1) - 2ky_1 + k^2 - R^2 = 0$$

Recall that the quadratic **e**quation has the following form:

$$aX^2 + bX + c = 0 \qquad (3)$$

Substitute (2) into (3) and factoring results in

$$(1 + m^2)x^2 - 2(h + m^2x_1 - my_1 + mk)x + h^2 + k^2 + y_1{}^2 + m^2x_1^2 - R^2$$
$$-2mx_1y_1 + 2kmx_1 - 2ky_1 = 0 \qquad (4)$$

The solution is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (5)$$

where:

$$a = 1 + m^2$$
$$b = -2(h + m^2x_1 - my_1 + mk) \qquad (6)$$
$$c = h^2 + k^2 + y_1{}^2 + m^2x_1^2 - R^2 - 2mx_1y_1 + 2kmx_1 - 2ky_1$$

Example: If the offset to the center of the circle with a radius of 2 units are defined a 3 units in the x-direction and 7 units in the y-direction (i.e., h = 3 and k = 7) and if a line passing through point (1, 4) has an azimuth of 62^o 11' 40", what are the coordinates of the intersection of this line with the circle (Hashimi, 1988).

Solution: Recall that the slope (m) is defined by the cotangent of the azimuth (m = cot Az) therefore the parameters a, b, and c can be found using the relationships shown in (6).

$$a = 1 + \cot^2 Az = 1 + \cot^2 62^o 11'40''$$
$$= 1.27811$$

$$b = -2\left(h + m^2 x_1 - m y_1 + mk\right)$$
$$b = -2\left(3 + \left(\cot^2 62^\circ 11'40''\right)(1) - \left(\cot 62^\circ 11'40''\right)(4) + \left(\cot 62^\circ 11'40''\right)(7)\right)$$
$$b = -9.72041$$

$$c = h^2 + k^2 + y_1^{\ 2} + m^2 x_1^2 - R^2 - 2m x_1 y_1 + 2km x_1 - 2k y_1$$
$$c = 3^2 + 7^2 + 4^2 + \left(\cot^2 62^\circ 11'40''\right)(1) - 2\left(\cot 62^\circ 11'40''\right)(1)(4) - 2^2 + 2\left(\cot 62^\circ 11'40''\right)(1)(7) - 2(4)(7)$$
$$c = -17.4423$$

Solving the equations by using the quadratic equation results in two sets of coordinates. For the x-coordinates, the intersections are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-9.72041 + \sqrt{-9.72041^2 - 4(1.27811)(17.4423)}}{2\,(1.28811)} = \underline{4.7044}$$

$$= \frac{-9.72041 - \sqrt{-9.72041^2 - 4(1.27811)(17.4423)}}{2\,(1.28811)} = \underline{2.9009}$$

Finally, substitute these values of x into the equation y = m(x - x_1) + y_1 thus giving the other coordinate value shown as:

$$y = m\left(x - x_1\right) + y_1 = \left(\cot 62^\circ 11'40''\right)(4.7044 - 1) + 4 = \underline{5.9536}$$
$$= \left(\cot 62^\circ 11'40''\right)(2.9009 - 1) + 4 = \underline{5.0025}$$

This approach is best performed using programmable calculators/computers. An alternative approach is to solve this problem by solving simple triangles. The problem is shown in figure 1. O is the center of the circle and points P_1 and P_2 are the two points of intersection. The line passes through point A and, naturally, P_1 and P_2.

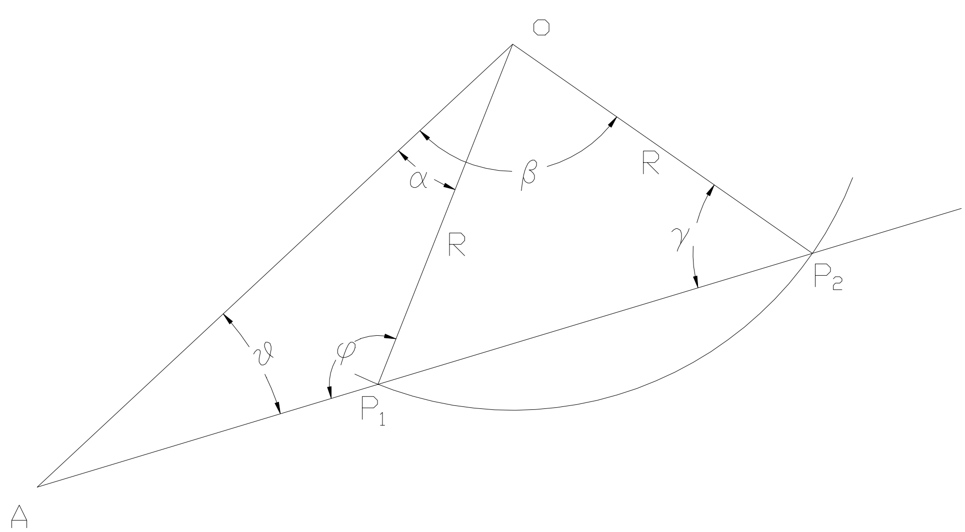


Figure 1. Line-circle intersection problem.

The distance AO is found using the Pythagorean Theorem

$$\begin{aligned} D_{AO} &= \sqrt{\left(X_O - X_A\right)^2 + \left(Y_O - Y_A\right)^2} \\ &= \sqrt{\left(3-1\right)^2 + \left(7-4\right)^2} \\ &= 3.606 \end{aligned}$$

The azimuth between A and O is found to be:

$$\begin{aligned} AZ_{AO} &= \tan^{-1}\left[\frac{X_O - X_A}{Y_O - Y_A}\right] = \tan^{-1}\left[\frac{3-1}{7-4}\right] \\ &= 33^o\ 41'\ 24'' \end{aligned}$$

The azimuth from A to P_1 is given as 62^o 11' 40". Also, the azimuth from A to P_2 is equal to the azimuth from A to P_1. Therefore, the angle at point A (θ) is the difference between the azimuths along the line and the azimuth from A to O.

$$\begin{aligned} \theta &= \left(62^o\ 11'\ 40''\right) - \left(33^o\ 41'\ 24''\right) \\ &= 28^o\ 30'16'' \end{aligned}$$

Also, from the problem statement, we know that $D_{OP1} = D_{OP2}$ = Radius of the circle = 2. Then, from the sine law given as

$$\frac{D_{OP_1}}{\sin\theta} = \frac{D_{AO}}{\sin\varphi} = \frac{D_{AO}}{\sin\gamma}$$

These relationships can be rearranged and relationships for φ and γ can be developed.

$$\begin{aligned} \varphi &= \sin^{-1}\left[\left(\frac{\sin\theta}{D_{OP_1}}\right)D_{AO}\right] = \sin^{-1}\left[\left(\frac{\sin 28^o 30'16''}{2}\right)3.606\right] \\ &= 120^o\ 38'\ 02'' \\ \gamma &= \sin^{-1}\left[\left(\frac{\sin\theta}{D_{OP_1}}\right)D_{AO}\right] = \sin^{-1}\left[\left(\frac{\sin 28^o 30'16''}{2}\right)3.606\right] \\ &= 59^o\ 21'57'' \end{aligned}$$

With φ and γ known, α and β can be found using the relationship that a triangle contains 180^o.

$$\alpha = 180^o - \left(120^o 38'02'' + 28^o 30'16''\right)$$
$$\alpha = 30^o\ 51'\ 42''$$

$$\beta = 180^p - \left(59^o 21'57'' + 28^o 30'16''\right)$$
$$\beta = 92^o\ \ 07'\ 47''$$

Then, compute the distance from A to P_1 and P_2 by using the relationships from the sine law:

$$\frac{D_{AP_1}}{\sin\alpha} = \frac{D_{AO}}{\sin\varphi} \qquad ; \qquad \frac{D_{AP_2}}{\sin\beta} = \frac{D_{AO}}{\sin\gamma}$$

from which the distances are found as

$$D_{AP_1} = \left(\frac{D_{AO}}{\sin\varphi}\right)\sin\alpha = \left(\frac{3.606}{\sin 120^o 38'02''}\right)\sin 30^o 51'42''$$
$$D_{AP_1} = 2.150$$

$$D_{AP_2} = \left(\frac{D_{AO}}{\sin\gamma}\right)\sin\beta = \left(\frac{3.606}{\sin 59^o 21'57''}\right)\sin 92^o 07'47''$$
$$D_{AP_2} = 4.188$$

The X and Y coordinates of the points of intersection can be computed as normal.

$$X_{P_1} = X_A + D_{AP_1}\sin AZ_{AP_1} = 1 + 2.150\sin 62^O 11'40'' = \underline{\underline{2.90}}$$
$$Y_{P_1} = Y_A + D_{AP_1}\cos AZ_{AP_1} = 1 + 2.149\cos 62^O 11'40'' = \underline{\underline{5.00}}$$

$$X_{P_{12}} = X_A + D_{AP_2}\sin AZ_{AP_1} = 1 + 4.188\sin 62^O 11'40'' = \underline{\underline{4.70}}$$
$$Y_{P_2} = Y_A + D_{AP_2}\cos AZ_{AP_1} = 1 + 4.188\cos 62^O 11'40'' = \underline{\underline{5.95}}$$

Check by inversing between the points of intersection (P_1 and P_2) and the center of the circle (O). These values should equal the radius of the circle, which is known.

$$D_{OP_1} = \sqrt{(2.90 - 3)^2 + (5.00 - 7)^2} = 2.00 \quad \surd$$
$$D_{OP_2} = \sqrt{(4.70 - 3)^2 + (5.95 - 7)^2} = 2.00 \quad \surd$$

Both values check to the two decimal place level. They may not agree perfectly with the known value of the radius because of rounding of the numbers in intermediate steps.

For another example of how to solve the line circle intersections problem, look at the appendix. This example was solved using Mathcad. The geometry is shown in figure 2.

REFERENCE

Hashimi, S., 1988. "SUR 221 Surveying Calculations Lecture Notes", Ferris State University, 90p.

APPENDIX

The following is an example of solving a line-circle intersection problem using Mathcad.

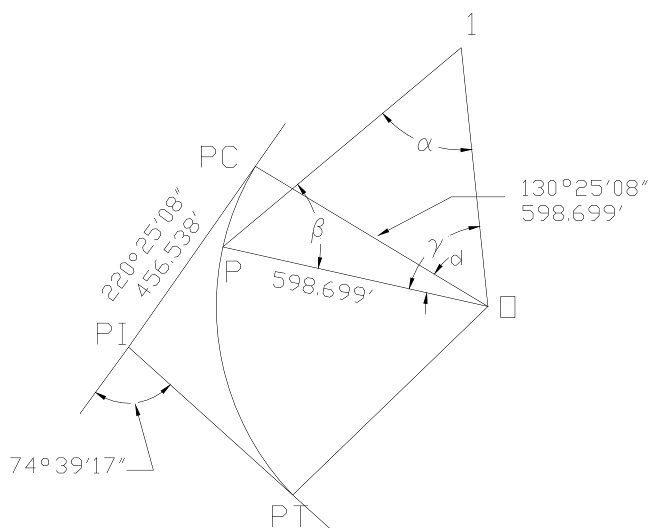


Figure 2. Line Intersection example using Mathcad.

Line-Circle Intersection

Given the following data, compute the
- coordinates of point P
- Coordinates of the PI
- Coordinates of the center of the circle (O)
- Arc length from the PC to point P
- Central angle from the PC to point P

Coordinates of point 1 on the line and the point on curve (PC):

$X1 := 5946.976$ $Y1 := 4801.176$

$XPC := 5429.259$ $YPC := 4447.866$

Azimuth from point 1 to the point of intersection:

$AZ\_1P := 231.3121$ (In ddd.mmss format) $a12 := \text{floor}(AZ\_1P)$ $amn := (AZ\_1P - a12) \cdot 100$

$$A1P := a12 + \frac{(\text{floor}(amn))}{60} + \left[(amn - \text{floor}(amn)) \cdot \frac{100}{3600}\right]$$

$A1P = 231.5225$ (in decimal degrees format)

Forward azimuth of the back tangent of the curve:

$AZ\_BT := 220.250$ (In ddd.mmss format) $a12 := \text{floor}(AZ\_BT)$ $amn := (AZ\_BT - a12) \cdot 100$

$$ABT := a12 + \frac{(\text{floor}(amn))}{60} + \left[(amn - \text{floor}(amn)) \cdot \frac{100}{3600}\right]$$

$ABT = 220.418889$ (in decimal degrees format)

Radius of curve: $Rad := 598.699$

Central angle of the curve (Δ):

$CA := 74.3917$ (In ddd.mmss format) $a12 := \text{floor}(CA)$ $amn := (CA - a12) \cdot 100$

$$\Delta := a12 + \frac{(\text{floor}(amn))}{60} + \left[(amn - \text{floor}(amn)) \cdot \frac{100}{3600}\right]$$

$\Delta = 74.655$ (in decimal degrees format)

$$rd := \frac{180}{\pi}$$

SOLUTION

Tangent Distance: $T := Rad \cdot \tan\left(\frac{\Delta}{2 \cdot rd}\right)$

$T = 456.538$

Azimuth from PC to Center of Curve: $AZpco := ABT - 90$

$AZpco = 130.41889$

Coordinates of center of circle

$$XO := XPC + Rad \cdot \sin\left(\frac{AZpco}{rd}\right) \qquad YO := YPC + Rad \cdot \cos\left(\frac{AZpco}{rd}\right)$$

$$XO = 5.8850633 \cdot 10^3 \qquad YO = 4.059687 \cdot 10^3$$

Coordinates of the PI:

$$XPI := XPC + T \cdot \sin\left(\frac{ABT}{rd}\right) \qquad YPI := XPC + T \cdot \cos\left(\frac{ABT}{rd}\right)$$

$$XPI = 5.1332528 \cdot 10^3 \qquad YPI = 5.0816851 \cdot 10^3$$

$$num := XO - X1$$
$$den := YO - Y1$$

The azimuth between point 1 on the line and the center of the circle is computed using the arc tangent function as

$$AZ := \operatorname{atan}\left(\frac{num}{den}\right)$$

$$AZ = 0.0833046$$

$$AZ1O := AZ + \pi$$

$$\alpha := A1P - \left(AZ1O \cdot \frac{180}{\pi}\right)$$

$$\alpha = 46.7494985$$

$$D1O := \sqrt{(XO - X1)^2 + (YO - Y1)^2}$$

$$D1O = 744.069$$

From the sine law, the the β can be computed as:

$$\beta := \operatorname{asin}\left[\left(\frac{\sin\left(\frac{\alpha}{rd}\right)}{Rad}\right) \cdot D1O\right]$$

$$\beta = 1.1318961$$

Then, the angle γ is found by subtracting the sum of the other two angles in the triangle, α and β

$$\gamma := 180 - \alpha - \beta \cdot \frac{180}{\pi}$$

$$\gamma = 68.3976341$$

The azimuth between the center of the circle (O) and the point of intersection (P) is:

$$AZOP := (AZ1O - \pi) - \left(\gamma \cdot \frac{\pi}{180}\right) + 2 \cdot \pi$$

$$AZOP = 5.172726$$

The coordinates of the point of intersection are now computed as normal:

$$XP := XO + Rad \cdot \sin(AZOP) \qquad YP := YO + Rad \cdot \cos(AZOP)$$

$$XP = 5.3486871 \cdot 10^3 \qquad YP = 4.325659 \cdot 10^3$$

The central angle between the PC to point P is

$$d := AZpco \cdot \left(\frac{\pi}{180}\right) + \pi - AZOP$$

$d = 0.245$

$$ang := d \cdot \frac{180}{\pi}$$

$ang = 14.0435214$ $\quad degs := floor(ang)$ $\quad minu := (ang - degs) \cdot 60$

Central angle deg, min, sec:

$mins := floor(minu)$

$degs = 14$

$mins = 2$

$secu = 36.677$

$secu := (minu - mins) \cdot 60$

Chord distance between the PC and the point P

$$c := \sqrt{(XPC - XP)^2 + (YPC - YP)^2}$$

$c = 146.3774751$

$$AZPPC := atan\left[\frac{(XPC - XP)}{(YPC - YP)}\right]$$

$AZPPC = 0.5828898$

$$ang := AZPPC \cdot \frac{180}{\pi}$$

$ang = 33.3971282$ $\quad degs := floor(ang)$ $\quad minu := (ang - degs) \cdot 60$

Azimuth from P to PC deg, min, sec:

$mins := floor(minu)$

$degs = 33$

$mins = 23$

$secu = 49.661$

$secu := (minu - mins) \cdot 60$

$$D := \left[\frac{360}{(2 \cdot \pi \cdot Rad)}\right] \cdot 100 \qquad l := 100 \cdot \frac{\Delta}{D}$$

The arc length from the PC to the point P is

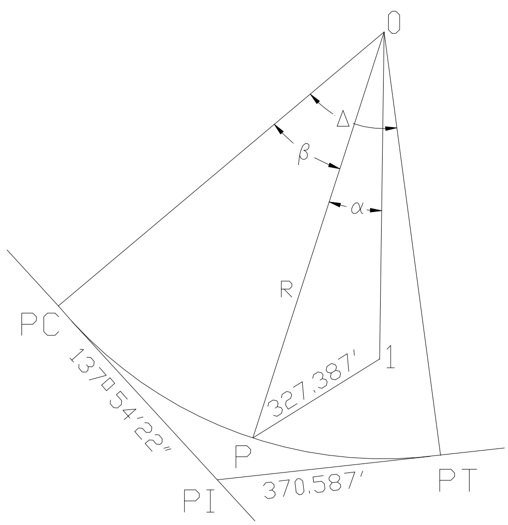
$$Arc := \left[\frac{l}{(\Delta)} \cdot d \cdot \left(\frac{180}{\pi}\right)\right]$$

$Arc = 146.745$

Example 2: Given: X_1 = 5313.674 Y_1 = 4200.812
X_{PI} = 4977.455 Y_{PI} = 3951.449
Distance from 1 to point P = 327.387'
Forward Azimuth of Back Tangent = 137^o 54' 22"
Radius of the circular curve = 819.524'
Central angle (Δ) = 48^o 39' 53"

Compute:
- The coordinates of point P
- Coordinates of the PC
- Coordinates of the center of the circle (O)
- Arc length from the PC to point P
- Distance and azimuth from the PC to P, and
- Central angle from the PC to point P

Solution:



Solution to parts of the horizontal circle:

$$T = R\tan\left(\frac{\Delta}{2}\right) = 819.524\tan\left(\frac{48^o39'53''}{2}\right)$$
$$= 370.587'$$

$$X_{PC} = X_{PI} + T\sin AZ_{PI-PC} = 4977.455' + 370.587\sin 317^o54'22''$$
$$X_{PC} = 4729.033'$$

$$Y_{PC} = Y_{PI} + T\cos AZ_{PI-PC} = 3951.449' + 370.587\cos 317^p54'22''$$
$$Y_{PC} = 4226.442'$$

$$AZ_{PC-O} = AZ_{PC-PI} - 90^o = \left(137^o 54'22''\right) - 90^o$$
$$= 47^o 54'22''$$

$$X_O = X_{PC} + R\sin AZ_{PC-O} = 4729.033' + 819.524\sin 47^o 54'22''$$
$$X_O = 5337.159'$$

$$Y_O = Y_{PC} + R\cos AZ_{PC-O} = 3951.449' + 819.524\cos 47^o 54'22''$$
$$Y_O = 4775.808'$$

$$D_{O-1} = \sqrt{\left(X_1 - X_O\right)^2 + \left(Y_1 - Y_O\right)^2}$$
$$= \sqrt{\left(5313.674 - 5337.159\right)^2 + \left(4200.812 - 4775.808\right)^2}$$
$$= 575.475'$$

$$\cos\alpha = \frac{D_{O-P}^2 + D_{O-1}^2 - D_{1-P}^2}{2D_{O-P}D_{O-1}} = \frac{819.524^2 + 575.475^2 - 327.387^2}{2\left(819.524\right)\left(575.475\right)}$$
$$\alpha = 18^o\ 17'\ 03''$$

$$AZ_{O-1} = \tan^{-1}\left[\frac{X_1 - X_O}{Y_1 - Y_O}\right] = \tan^{-1}\left[\frac{5313.674 - 5337.159}{4200.812 - 4775.808}\right]$$
$$AZ_{O-1} = 182^o\ 20'\ 20''$$

$$AZ_{O-P} = AZ_{O-1} + \alpha = \left(182^o\ 20'20''\right) + \left(18^o\ 17'03''\right)$$
$$AZ_{O-P} = 200^o\ 37'23''$$

$$X_P = X_O + D_{O-P}\sin AZ_{O-P} = 5337.159 + 819.524\sin 200^o\ 37'23''$$
$$X_P = 5048.508'$$

$$Y_P = Y_O + D_{O-P}\cos AZ_{O-P} = 4775.808 + 819.524\cos 200^o\ 37'23''$$
$$Y_P = 4008.801'$$

$$D = \left(\frac{360^o}{2\pi R}\right) = \left(\frac{360^o}{2\pi(819.524)}\right)$$

$$D = 6^o\ 59'\ 29''$$

$$L = 100\left(\Delta/D\right) = 100\left(\frac{48^o 39'53''}{6^o 59'29''}\right)$$

$$L = 696.07'$$

$$\beta = AZ_{O-PC} - AZ_{O-P} = \left(227^o 54'22''\right) - \left(200^o 37'23''\right)$$

$$= 27^o\ 16'\ 59'' = d$$

$$\frac{Arc_{PC-P}}{d} = \frac{L}{\Delta} \Rightarrow Arc_{PC-P} = \left(\frac{696.07}{48^o 39'53''}\right) 27^o 16'59''$$

$$= 390.240'$$

$$D_{PC-P} = \sqrt{\left(X_P - X_{PC}\right)^2 + \left(Y_P - Y_{PC}\right)^2}$$

$$= \sqrt{(5048.508 - 4729.033)^2 + (4008.801 - 4226.442)^2}$$

$$= 386.564'$$

$$AZ_{PC-P} = \tan^{-1}\left[\frac{X_P - X_{PC}}{Y_P - Y_{PC}}\right] = \tan^{-1}\left[\frac{5048.508 - 4729.033}{4008.801 - 4226.442}\right]$$

$$= 124^o\ 15'\ 52''$$