LINE-CIRCLE INTERSECTION

Surveying Engineering

Ferris State University

LINE-CIRCLE INTERSECTION

- Write equation of line and circle

$$y - y_1 = m(x - x_1) \rightarrow y = m(x - x_1) + y_1$$

$$R^2 = (x - h)^2 + (y - k)^2$$

- Expand equation of circle

$$(x - h)^2 + (y - k)^2 - R^2 = 0$$

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 - R^2 = 0$$

LINE-CIRCLE INTERSECTION

- Substitute into equation of line written in terms of y

$$x^2 - 2hx + h^2 + [m(x - x_1) + y_1]^2 - 2k[m(x - x_1) + y_1] + k^2 - R^2 = 0$$
$$x^2 - 2hx + h^2 + m^2x^2 - 2m^2xx_1 + 2mxy_1 + m^2x_1^2 - 2mx_1y_1 + y_1^2$$
$$-2kmx + 2kmx_1) - 2ky_1 + k^2 - R^2 = 0$$

- Has form of quadratic equation

$$(1 + m^2)x^2 - 2(h + m^2x_1 - my_1 + mk)x + h^2 + k^2 + y_1^2 + m^2x_1^2 - R^2$$
$$-2mx_1y_1 + 2kmx_1 - 2ky_1 = 0$$

LINE-CIRCLE INTERSECTION

- Solution of quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- where:

$$a = 1 + m^2$$
$$b = -2(h + m^2x_1 - my_1 + mk)$$
$$c = h^2 + k^2 + y_1^2 + m^2x_1^2 - R^2 - 2mx_1y_1 + 2kmx_1 - 2ky_1$$

LINE-CIRCLE INTERSECTION EXAMPLE

- R = 2; h = 3; k = 7; x_A, y_A of point on line= (1,4); Az = 62°11'40"
- Solution: recall m is slope defined by cotangent of azimuth

LINE-CIRCLE INTERSECTION EXAMPLE

$$a = 1 + \cot^2 Az = 1 + \cot^2 62^o 11' 40"$$
$$a = 1.27811$$

$$b = -2\left(h + m^2 x_1 - m y_1 + mk\right)$$
$$b = -2\left(3 + \left(\cot^2 62^o 11' 40"\right)(1) - \left(\cot 62^o 11' 40"\right)(4) + \left(\cot 62^o 11' 40"\right)(7)\right)$$
$$b = -9.72041$$

$$c = h^2 + k^2 + {y_1}^2 + m^2 x_1^2 - R^2 - 2 m x_1 y_1 + 2 k m x_1 - 2 k y_1$$
$$c = 3^2 + 7^2 + 4^2 + \left(\cot^2 62^o 11' 40"\right)(1) - 2\left(\cot 62^o 11' 40"\right)(1)(4) - 2^2 + 2\left(\cot 62^o 11' 40"\right)(1)(7) - 2(4)(7)$$
$$c = -17.4423$$

LINE-CIRCLE INTERSECTION EXAMPLE

- Solve using quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-9.72041 + \sqrt{-9.72041^2 - 4(1.27811)(17.4423)}}{2\,(1.28811)} = \underline{4.7044}$$

$$= \frac{-9.72041 - \sqrt{-9.72041^2 - 4(1.27811)(17.4423)}}{2\,(1.28811)} = \underline{2.9009}$$

LINE-CIRCLE INTERSECTION EXAMPLE

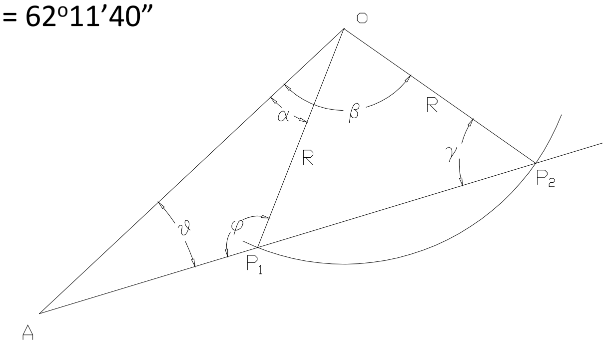
- Solve for y:

$$y = m\,(x - x_1) + y_1 = (\cot\, 62^{o}11'40'')(4.7044 - 1) + 4 = \underline{5.9536}$$

$$= (\cot\, 62^{o}11'40'')(2.9009 - 1) + 4 = \underline{5.0025}$$

LINE-CIRCLE INTERSECTION EXAMPLE - TRIANGLE APPROACH

- Given:
  - R = 2; h = 3; k = 7; x_A, y_A of point on line= (1,4)
  - Az = 62°11'40"



LINE-CIRCLE INTERSECTION EXAMPLE - TRIANGLE APPROACH

- Distance and Azimuth from A to center of circle found by:

$$D_{AO} = \sqrt{(X_O - X_A)^2 + (Y_O - Y_A)^2}$$
$$= \sqrt{(3-1)^2 + (7-4)^2}$$
$$= 3.606$$
$$AZ_{AO} = \tan^{-1}\left[\frac{X_O - X_A}{Y_O - Y_A}\right] = \tan^{-1}\left[\frac{3-1}{7-4}\right]$$
$$= 33^o\,41'24''$$

- Angle at point A is difference in azimuths along the line and from A to O

$$\theta = \left(62^o\,11'40''\right) - \left(33^o\,41'24''\right)$$
$$= 28^o\,30'16''$$

LINE-CIRCLE INTERSECTION EXAMPLE - TRIANGLE APPROACH

- Using sine law, φ and γ, are found as:

$$\varphi = \sin^{-1}\left[\left(\frac{\sin\theta}{D_{OP_1}}\right)D_{AO}\right] = \sin^{-1}\left[\left(\frac{\sin 28^o 30'16''}{2}\right)3.606\right]$$

$$= 120^o\ 38'02''$$

$$\gamma = \sin^{-1}\left[\left(\frac{\sin\theta}{D_{OP_1}}\right)D_{AO}\right] = \sin^{-1}\left[\left(\frac{\sin 28^o 30'16''}{2}\right)3.606\right]$$

$$= 59^o\ 21'57''$$

LINE-CIRCLE INTERSECTION EXAMPLE - TRIANGLE APPROACH

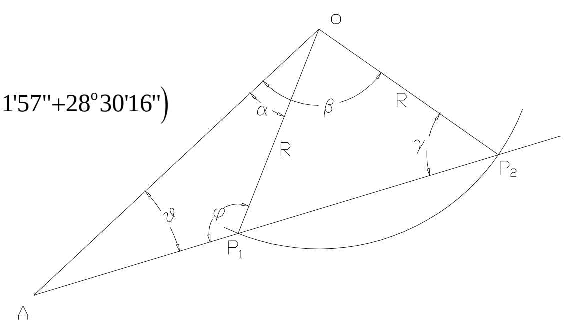
- From this:

$$\alpha = 180^o - \left(120^o 38'02'' + 28^o 30'16''\right)$$

$$\alpha = 30^o\ 51'\ 42''$$

$$\beta = 180^p - \left(59^o 21'57'' + 28^o 30'16''\right)$$

$$\beta = 92^o\ 07'\ 47''$$



LINE-CIRCLE INTERSECTION EXAMPLE - TRIANGLE APPROACH

- Using the sine law again, the distances from point A to points 1 and 2 are:

$$D_{AP_1} = \left(\frac{D_{AO}}{\sin\varphi}\right)\sin\alpha = \left(\frac{3.606}{\sin 120^o 38'02''}\right)\sin 30^o 51'42''$$

$$D_{AP_1} = 2.150$$

$$D_{AP_2} = \left(\frac{D_{AO}}{\sin\gamma}\right)\sin\beta = \left(\frac{3.606}{\sin 59^o 21'57''}\right)\sin 92^o 07'47''$$

$$D_{AP_2} = 4.188$$

LINE-CIRCLE INTERSECTION EXAMPLE - TRIANGLE APPROACH

- The coordinates of the points of intersection

$$X_{P_1} = X_A + D_{AP_1}\sin AZ_{AP_1} = 1 + 2.150\sin 62^O 11'40'' = \underline{\underline{2.90}}$$

$$Y_{P_1} = Y_A + D_{AP_1}\cos AZ_{AP_1} = 1 + 2.149\cos 62^O 11'40'' = \underline{\underline{5.00}}$$

$$X_{P_{12}} = X_A + D_{AP_2}\sin AZ_{AP_1} = 1 + 4.188\sin 62^O 11'40'' = \underline{\underline{4.70}}$$

$$Y_{P_2} = Y_A + D_{AP_2}\cos AZ_{AP_1} = 1 + 4.188\cos 62^O 11'40'' = \underline{\underline{5.95}}$$

LINE-CIRCLE INTERSECTION EXAMPLE - TRIANGLE APPROACH

- As a check, compute distances from the center of circle to the points of intersection (1 and 2)
  - Should be equal to radius of circle

$$D_{OP_1} = \sqrt{(2.90 - 3)^2 + (5.00 - 7)^2} = 2.00 \quad \surd$$

$$D_{OP_2} = \sqrt{(4.70 - 3)^2 + (5.95 - 7)^2} = 2.00 \quad \surd$$

EXAMPLE #2

$X_1 = 5313.674$ $\quad Y_1 = 4200.812$

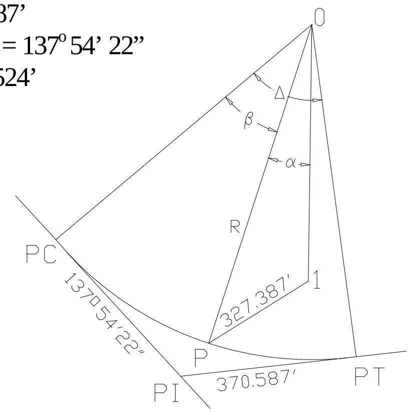
$X_{PI} = 4977.455$ $\quad Y_{PI} = 3951.449$

Distance from 1 to point P = 327.387’

Forward Azimuth of Back Tangent = 137° 54’ 22”

Radius of the circular curve = 819.524’

Central angle (Δ) = 48° 39’ 53”



EXAMPLE #2

- Solving part of horizontal circle

$$T = R\tan\left(\frac{\Delta}{2}\right) = 819.524\tan\left(\frac{48^o39'53''}{2}\right)$$
$$= 370.587'$$

$$X_{PC} = X_{PI} + T\sin AZ_{PI-PC} = 4977.455' + 370.587\sin 317^o54'22''$$
$$X_{PC} = 4729.033'$$

$$Y_{PC} = Y_{PI} + T\cos AZ_{PI-PC} = 3951.449' + 370.587\cos 317^p54'22''$$
$$Y_{PC} = 4226.442'$$

EXAMPLE #2

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$$AZ_{PC-O} = AZ_{PC-PI} - 90^o = \left(137^o54'22''\right) - 90^o$$
$$= 47^o54'22''$$

$$X_O = X_{PC} + R\sin AZ_{PC-O} = 4729.033' + 819.524\sin 47^o54'22''$$
$$X_O = 5337.159'$$

$$Y_O = Y_{PC} + R\cos AZ_{PC-O} = 3951.449' + 819.524\cos 47^o54'22''$$
$$Y_O = 4775.808'$$

EXAMPLE #2

$$D_{O-1} = \sqrt{(X_1 - X_O)^2 + (Y_1 - Y_O)^2}$$
$$= \sqrt{(5313.674 - 5337.159)^2 + (4200.812 - 4775.808)^2}$$
$$= 575.475'$$

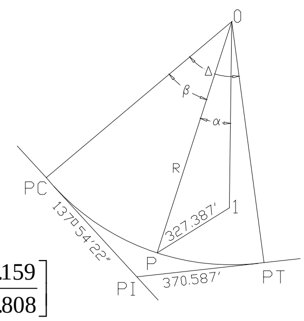
$$\cos\alpha = \frac{D_{O-P}^2 + D_{O-1}^2 - D_{1-P}^2}{2D_{O-P}D_{O-1}} = \frac{819.524^2 + 575.475^2 - 327.387^2}{2(819.524)(575.475)}$$

$$\alpha = 18^\circ\ 17'\ 03''$$

$$AZ_{O-1} = \tan^{-1}\left[\frac{X_1 - X_O}{Y_1 - Y_O}\right] = \tan^{-1}\left[\frac{5313.674 - 5337.159}{4200.812 - 4775.808}\right]$$

$$AZ_{O-1} = 182^\circ\ 20'\ 20''$$

$$AZ_{O-P} = AZ_{O-1} + \alpha = (182^\circ\ 20'20'') + (18^\circ\ 17'03'')$$
$$AZ_{O-P} = 200^\circ\ 37'23''$$



EXAMPLE #2

- The coordinates of the point of intersection between the circle and line is

$$X_P = X_O + D_{O-P}\sin AZ_{O-P} = 5337.159 + 819.524\sin 200^\circ\ 37'23''$$
$$X_P = 5048.508'$$

$$Y_P = Y_O + D_{O-P}\cos AZ_{O-P} = 4775.808 + 819.524\cos 200^\circ\ 37'23''$$
$$Y_P = 4008.801'$$

EXAMPLE #2

$$D = \left(\frac{360^o}{2\pi R}\right) = \left(\frac{360^o}{2\pi(819.524)}\right)$$
$$D = 6^o\ 59'\ 29''$$

$$L = 100\left(\Delta/D\right) = 100\left(\frac{48^o 39'53''}{6^o 59'29''}\right)$$
$$L = 696.07'$$

$$\beta = AZ_{O-PC} - AZ_{O-P} = \left(227^o 54'22''\right) - \left(200^o 37'23''\right)$$
$$= 27^o\ 16'\ 59'' = d$$

EXAMPLE #2

$$\frac{Arc_{PC-P}}{d} = \frac{L}{\Delta} \Rightarrow Arc_{PC-P} = \left(\frac{696.07}{48^o 39'53''}\right) 27^o 16'59''$$
$$= 390.240'$$

$$D_{PC-P} = \sqrt{\left(X_P - X_{PC}\right)^2 + \left(Y_P - Y_{PC}\right)^2}$$
$$= \sqrt{\left(5048.508 - 4729.033\right)^2 + \left(4008.801 - 4226.442\right)^2}$$
$$= 386.564'$$

$$AZ_{PC-P} = \tan^{-1}\left[\frac{X_P - X_{PC}}{Y_P - Y_{PC}}\right] = \tan^{-1}\left[\frac{5048.508 - 4729.033}{4008.801 - 4226.442}\right]$$
$$= 124^o\ 15'\ 52''$$