

5

CIRCLE-CIRCLE INTERSECTION

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INTRODUCTION

The circle-circle intersection problem involves the solution of two quadratic equations (Hashimi, 1988). Writing the equation of a circle for both circles, we have

$$(X - h_1)^2 + (Y - k_1)^2 - R_1^2 = 0$$

$$(X - h_2)^2 + (Y - k_2)^2 - R_2^2 = 0$$

where h_i is the x-coordinate of the center of the circle and k_i is the y-coordinate. Lets designate the following

$$u = R_1^2 - R_2^2 + h_1^2 + h_2^2 - 2h_1h_2 - k_1^2 + k_2^2$$

$$v = h_2 - h_1$$

$$w = k_1 - k_2$$

Then, expanding the equations defining the circles and solving them simultaneously, we have the following quadratic form

$$4(w^2 + v^2)Y^2 + 4(uw - 2v^2k_1)Y + u^2 - 4v^2(R_1^2 - k_1^2) = 0$$

The solution is through the quadratic equation

$$Y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where,

$$a = 4(w^2 + v^2)$$

$$b = 4(uw - 2v^2k_1)$$

$$c = u^2 - 4v^2(R_1^2 - k_1^2)$$

An example using this method is shown in the following MathCAD program.

CIRCLE-CIRCLE INTERSECTION PROBLEM

$$h_1 := 5 \quad k_1 := 7 \quad R_1 := 2$$

$$h_2 := 8 \quad k_2 := 3 \quad R_2 := 4$$

$$u := R_1^2 - R_2^2 + h_1^2 + h_2^2 - 2 \cdot h_1 \cdot h_2 - k_1^2 + k_2^2$$

$$v := h_2 - h_1$$

$$w := k_1 - k_2$$

$$a := 4 \cdot (w^2 + v^2)$$

$$b := 4 \cdot (u \cdot w - 2 \cdot v^2 \cdot k_1)$$

$$c := u^2 - 4 \cdot v^2 \cdot (R_1^2 - k_1^2)$$

First possible solution:

$$Y_1 := \frac{-b + \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

$$Y_1 = 6.87$$

$$d := (Y_1 - k_1)^2 - R_1^2$$

$$e := (Y_1 - k_2)^2 - R_2^2$$

$$X_1 := \frac{h_2^2 - h_1^2 - d + e}{2 \cdot v}$$

$$X_1 = 7.$$

Second possible solution:

$$Y_2 := \frac{-b - \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

$$Y_2 = 5.05$$

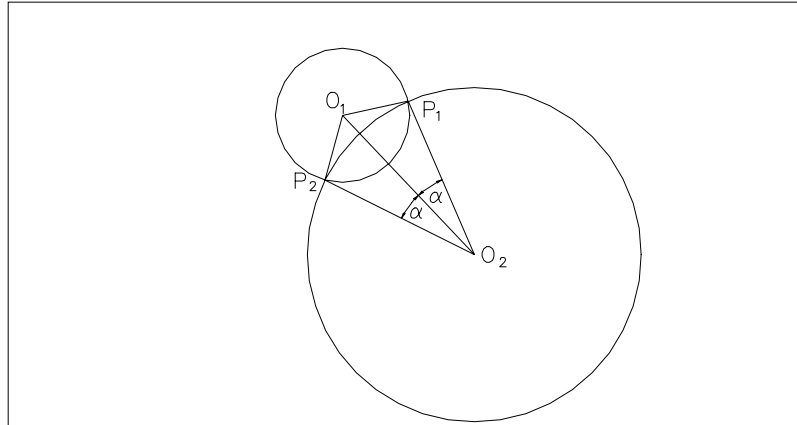
$$c := (Y_2 - k_1)^2 - R_1^2$$

$$d := (Y_2 - k_2)^2 - R_2^2$$

$$X_2 := \frac{h_2^2 - h_1^2 - c + d}{2 \cdot v}$$

$$X_2 = 4.56$$

Alternatively, one can use the triangle solution. This is best shown by looking at the example that was just used. First, compute the distance and azimuth between the centers of the two circles (shown as O_1 and O_2 in the figure).



The distance is found using the Pythagorean theorem while the azimuth of the line is found using the arctangent function.

$$\begin{aligned} D_{O_1O_2} &= \sqrt{(h_2 - h_1)^2 + (k_2 - k_1)^2} \\ &= \sqrt{(8 - 5)^2 + (3 - 7)^2} \\ &= 5 \end{aligned}$$

$$\begin{aligned} Az_{O_1O_2} &= \tan^{-1} \left[\frac{h_2 - h_1}{k_2 - k_1} \right] = \tan^{-1} \left[\frac{8 - 5}{3 - 7} \right] \\ &= 143^\circ 07' 48'' \end{aligned}$$

The law of sines can be written in a general form as: $a^2 = b^2 + c^2 - 2bc \cos A$. This can be rewritten as: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$. A is the angle with sides b and c being the adjacent sides and side a being the opposite length of angle A . Then, using the law of sines, solve for angle α .

$$\cos \alpha = \frac{D_{O_2P_1}^2 + D_{O_1O_2}^2 - D_{O_1P_1}^2}{2D_{O_2P_1}D_{O_1O_2}} = \frac{4^2 + 5^2 - 2^2}{2(4)(5)}$$

$$\alpha = 22^\circ 19' 54''$$

The azimuth from O_1 to the first point of intersection P_1 is found by adding α .

$$\begin{aligned} Az_{O_2P_1} &= Az_{O_2O_1} + \alpha = (323^\circ 07' 48'') + (22^\circ 19' 54'') \\ &= 345^\circ 27' 42'' \end{aligned}$$

With the azimuth and distance (this is the radius) known from the origin of circle 2, the X and Y coordinates are easily computed.

$$\begin{aligned} X_{P_1} &= X_{O_2} + R_2 \sin Az_{O_2P_1} = 8 + 4 \sin 345^\circ 27' 42'' \\ &= 7.00 \end{aligned}$$

$$\begin{aligned} Y_{P_1} &= Y_{O_2} + R_2 \cos Az_{O_2P_1} = 3 + 4 \cos 345^\circ 27' 42'' \\ &= 6.87 \end{aligned}$$

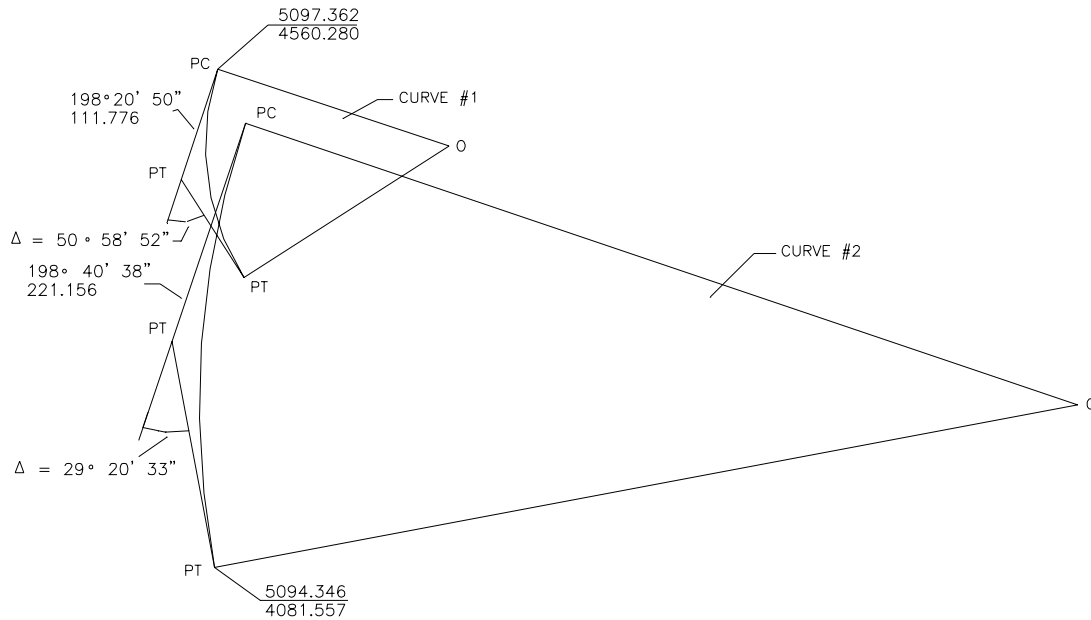
As a check, inverse between P_1 and O_1 . This should equal the radius of circle 1.

$$\sqrt{(X_{P_1} - X_{O_1})^2 + (Y_{P_1} - Y_{O_1})^2} = \sqrt{(7.00 - 5)^2 + (6.87 - 7)^2} = 2.00 = R_1$$

Since there are two points of intersection for this problem, we do the same for the triangle $O_1O_2P_2$.

Lets look at a more complex problem to the circle-circle intersection problem. This one involves the intersection of two horizontal curves. The following data is given:

Curve #1	$X_{PC} = 5097.362$	$Y_{PC} = 4560.280$
	Forward azimuth of back tangent: $198^\circ 20' 50''$	
	Tangent distance = 111.776	
	Central angle (Δ) = $50^\circ 58' 52''$	
	Curve is concave left	
Curve #2	$X_{PT} = 5094.346$	$Y_{PT} = 4081.557$
	Forward azimuth of back tangent: $198^\circ 40' 38''$	
	Tangent distance = 221.156	
	Central angle (Δ) = $29^\circ 20' 33''$	
	Curve is concave left	



The problem is to find the following:

- The coordinates of the point of intersection of the two circles
- The coordinates of the center of the two circles
- The arc length from the PCs of both circles to the point of intersection
- The distance and azimuth from the PCs of both circles to the point of intersection
- The central angle from both PCs to the point of intersection.

The first order is to compute the missing elements of both horizontal curves. For curve #1:

$$T = R \tan \frac{\Delta}{2} \Rightarrow R = \frac{T}{\tan \frac{\Delta}{2}} = \frac{111.776}{\tan \left(\frac{50^\circ 58' 52''}{2} \right)} = 234.4427$$

$$Az_{PC-O} = Az_{PC-PI} - 90^\circ = 198^\circ 20' 50'' - 90^\circ = 108^\circ 20' 50''$$

$$X_O = X_{PC} + R \sin Az_{PC-O} = 5097.362 + 234.4427 \sin(108^\circ 20' 50'') = 5319.8871$$

$$Y_O = Y_{PC} + R \cos Az_{PC-O} = 4560.280 + 234.4427 \cos(108^\circ 20' 50'') = 4486.4833$$

$$Az_{O-PT} = Az_{O-PC} - \Delta = (288^\circ 20' 50'') - (50^\circ 58' 52'') = 237^\circ 21' 58''$$

$$X_{PT} = X_O + R \sin Az_{O-PT} = 5319.8871 + 234.4427 \sin(237^\circ 21' 58'') = 5122.4550$$

$$Y_{PT} = Y_O + R \cos Az_{O-PT} = 4486.4833 + 234.4427 \cos(237^\circ 21' 58'') = 4360.0556$$

For curve #2:

$$R = \frac{T}{\tan \frac{\Delta}{2}} = \frac{221.156}{\tan\left(\frac{29^\circ 20' 33''}{2}\right)} = 844.7232$$

$$Az_{PC-O} = Az_{PC-PI} - 90^\circ = (198^\circ 40' 38'') - 90^\circ = 108^\circ 40' 38''$$

$$Az_{PI-PT} = Az_{pc-pi} - \Delta = (198^\circ 40' 38'') - (29^\circ 20' 33'') = 169^\circ 20' 05''$$

$$Az_{PT-O} = Az_{PT-PI} + 90^\circ = (349^\circ 20' 05'') + 90^\circ = 79^\circ 20' 05''$$

$$X_O = X_{PT} + R \sin Az_{PT-O} = 5094.346 + 844.7232 \sin(79^\circ 20' 05'') = 5924.4767$$

$$Y_O = Y_{PT} + R \cos Az_{PT-O} = 4081.557 + 844.7232 \cos(79^\circ 20' 05'') = 4237.8909$$

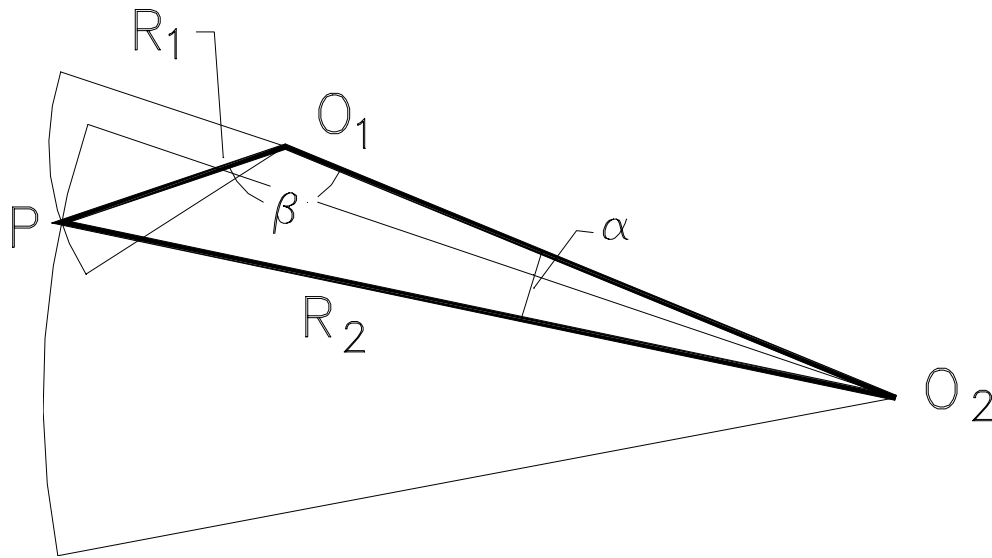
$$X_{PI} = X_{PT} + T \sin Az_{PT-PI} = 5094.346 + 221.156 \sin(349^\circ 20' 05'') = 5053.4164$$

$$Y_{PI} = Y_{PT} + T \cos Az_{PT-PI} = 4081.557 + 221.156 \cos(349^\circ 20' 05'') = 4298.8926$$

$$X_{PC} = X_{PI} + T \sin Az_{PI-PC} = 5053.4164 + 221.156 \sin(18^\circ 40' 38'') = 5124.2386$$

$$Y_{PC} = Y_{PI} + T \cos Az_{PI-PC} = 4298.8926 + 221.156 \cos(18^\circ 40' 38'') = 4508.4020$$

With the elements of both horizontal curves now known, we can look at the triangle formed by the origin of both horizontal curves and the point of intersection of the two arcs as shown in the next figure.



Then the distance between the two centers of the circles can be computed using the Pythagorean Theorem while the azimuth can be determined using the arctangent function.

$$\begin{aligned} D_{O_1O_2} &= \sqrt{(X_{O_2} - X_{O_1})^2 + (Y_{O_2} - Y_{O_1})^2} \\ &= \sqrt{(5924.4767 - 5319.8871)^2 + (4237.8909 - 4486.4833)^2} \\ &= 653.7024 \end{aligned}$$

$$\begin{aligned} Az_{O_1O_2} &= \tan^{-1} \left[\frac{X_{O_2} - X_{O_1}}{Y_{O_2} - Y_{O_1}} \right] = \tan^{-1} \left[\frac{5924.4767 - 5319.8871}{4237.8909 - 4486.4833} \right] \\ &= 112^\circ 21' 04.5'' \end{aligned}$$

The angle measured at the center of circle #2 is found using the cosine law.

$$\begin{aligned} \alpha &= \cos^{-1} \left[\frac{D_{O_1O_2}^2 + R_1^2 - R_2^2}{2 D_{O_1O_2} R_2} \right] = \cos^{-1} \left[\frac{(653.7024)^2 + (234.4427)^2 - (844.7232)^2}{2 (653.7024)(234.4427)} \right] \\ &= 138^\circ 58' 58.4'' \end{aligned}$$

The angle at the center of circle 1 (β) is also found using the cosine law.

$$\beta = \cos^{-1} \left[\frac{D_{O_1O_2}^2 + R_1^2 - R_2^2}{2D_{O_1O_2}R_2} \right] = \cos^{-1} \left[\frac{(653.7024)^2 + (234.4427)^2 - (844.7232)^2}{2(653.7024)(234.4427)} \right]$$

$$= 138^\circ 58' 58.4''$$

The coordinates of the point of intersection can now be determined after computing the azimuth from the origin of the second circle to the point of intersection.

$$Az_{O_2P} = Az_{O_2O_1} - \alpha = (292^\circ 21' 04.5'') - (10^\circ 29' 40.8'') = 281^\circ 51' 23.7''$$

$$X_P = X_{O_2} + R_2 \sin Az_{O_2P} = 5924.4767 + 844.7232 \sin (281^\circ 51' 23.7'') = \underline{5097.776}$$

$$Y_P = Y_{O_2} + R_2 \cos Az_{O_2P} = 4237.8909 + 844.7232 \cos (281^\circ 51' 23.7'') = \underline{4411.450}$$

As a check, compute the coordinates of the point of intersection from the center of circle #1. Again, the azimuth between O_1 and P must be determined first.

$$Az_{O_1P} = Az_{O_1O_2} + \beta = (112^\circ 21' 04.5'') + (138^\circ 58' 58.4'') = 251^\circ 20' 02.9''$$

$$X_P = X_{O_1} + R_1 \sin Az_{O_1P} = 5319.8871 + 234.4427 \sin (251^\circ 20' 02.9'') = \underline{5097.776} \quad (\text{check})$$

$$Y_P = Y_{O_1} + R_1 \cos Az_{O_1P} = 4486.4833 + 234.4427 \cos (251^\circ 20' 02.9'') = \underline{4411.450} \quad (\text{check})$$

Using basic principles from horizontal curve geometry the other desired elements of this problem can be found. The degree of curve (D) is computed using the arc-definition.

$$D_1 = \left(\frac{360^\circ}{2\pi R_1} \right) 100' = \left(\frac{360^\circ}{2\pi 234.4427'} \right) 100' = 24^\circ 26' 20.9''$$

$$D_2 = \left(\frac{360^\circ}{2\pi R_2} \right) 100' = \left(\frac{360^\circ}{2\pi 844.7232'} \right) 100' = 6^\circ 46' 58.0''$$

The central angle (δ) from the PC to point P is found as follows.

$$\delta_{PC_1P} = Az_{O_1PC_1} - Az_{O_1P} = (288^\circ 20' 50'') - (251^\circ 20' 02.9'') = 37^\circ 00' 47.1''$$

$$\delta_{PC_2P} = Az_{O_2PC_2} - Az_{O_2P} = (288^\circ 40' 38'') - (281^\circ 51' 23.7'') = 6^\circ 49' 14.3''$$

The chord distance (c) from the PC to point P can also be easily calculated.

$$c_{PC_1P} = 2 R_1 \sin\left(\frac{\delta_{PC_1P}}{2}\right) = 2(234.4427') \sin\left(\frac{37^\circ 00' 47.1''}{2}\right) = 148.830'$$

$$c_{PC_2P} = 2 R_2 \sin\left(\frac{\delta_{PC_2P}}{2}\right) = 2(844.7232') \sin\left(\frac{6^\circ 49' 14.3''}{2}\right) = 100.499'$$

The corresponding arc distance from the PC to P is computed using the following relationship.

$$c'_{PC_1P} = \frac{(\delta_{PC_1P})100'}{D_1} = \frac{(37^\circ 00' 47.1'')100'}{24^\circ 26' 20.9''} = 151.450'$$

$$c'_{PC_2P} = \frac{(\delta_{PC_2P})100'}{D_2} = \frac{(6^\circ 49' 14.3'')100'}{6^\circ 46' 58.0''} = 100.558'$$

As a rough check, the arc distance cannot be smaller than the corresponding chord distances. Finally, the azimuth from the PC to P is found by subtracting the deflection angle at the PC to the point P.

$$AZ_{PC_1P} = AZ_{PC_1PI_1} - \frac{\delta_{PC_1P}}{2} = (198^\circ 20' 50'') - \left(\frac{37^\circ 00' 47.1''}{2}\right) = 179^\circ 50' 26''$$

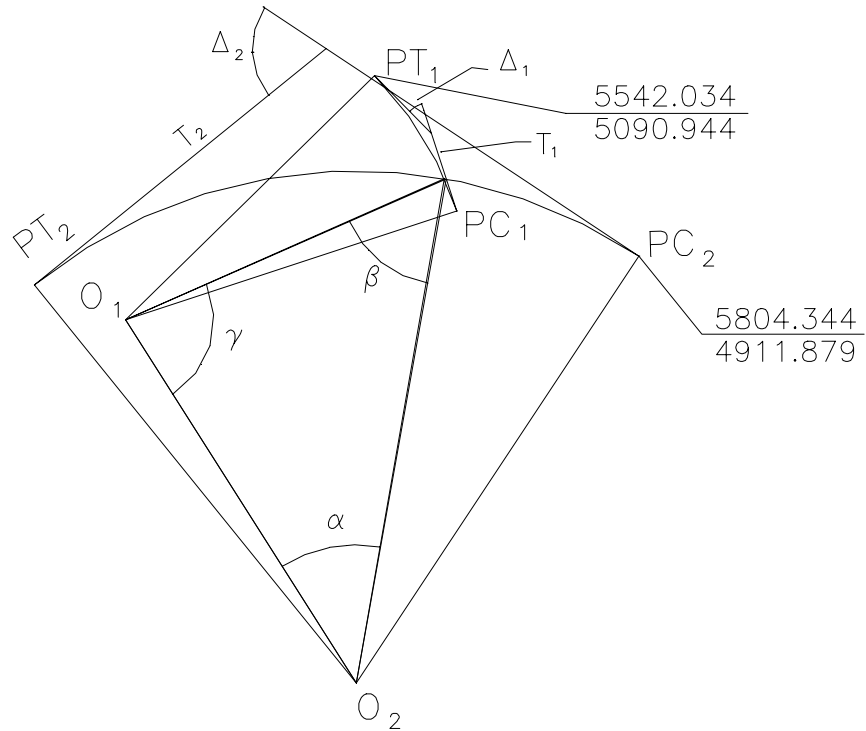
$$AZ_{PC_2P} = AZ_{PC_2PI_2} - \frac{\delta_{PC_2P}}{2} = (198^\circ 40' 38'') - \left(\frac{6^\circ 49' 14.3''}{2}\right) = 195^\circ 16' 01''$$

Lets look at another example that is similar to the last example. The following data is given:

Curve #1 $X_{PT} = 5542.034$ $Y_{PT} = 5090.944$
 Forward azimuth of back tangent: $341^\circ 49' 54''$
 Tangent distance = 80.751
 Central angle (Δ) = $26^\circ 15' 33''$
 Curve is concave left

Curve #2: $X_{PC} = 5804.344$ $Y_{PC} = 303^\circ 31' 06''$
 Tangent distance = 372.526
 Central angle (Δ) = $72^\circ 29' 00''$
 Curve is concave left

The diagram of this problem follows:



Recognizing that the radius of a horizontal curve is found using the relationship

$R = \frac{T}{\tan \frac{\Delta}{2}}$, then the radii of the two curves are calculated to be:

$$R_1 = \frac{80.751}{\tan\left(\frac{26^\circ 15' 33''}{2}\right)} = 346.1969$$

$$R_2 = \frac{372.526}{\tan\left(\frac{72^\circ 29' 00''}{2}\right)} = 508.2163$$

To compute the coordinates of the origin of circle #1, first compute the azimuth between the PT and the origin. Then the coordinates can be found using the formula to transform the polar coordinates to rectangular. These steps are as follows:

$$Az_{P_1, P_1} = Az_{P_1, P_1} - \Delta_1 = (341^\circ 49' 54'') - (26^\circ 15' 33'') = 315^\circ 34' 21''$$

$$Az_{P_1, O_1} = Az_{P_1, P_1} + 90^\circ = (315^\circ 34' 21'') + 90^\circ = 225^\circ 34' 21''$$

$$X_{O_1} = X_{PT_1} + R_1 \sin Az_{PT_1O_1} = 5542.034 + 346.1969 \sin (225^\circ 34' 21'') = 5294.8021$$

$$Y_{O_1} = Y_{PT_1} + R_1 \cos Az_{PT_1O_1} = 5090.944 + 346.1969 \cos (225^\circ 34' 21'') = 4848.6040$$

The coordinates of the origin of curve #2 can be computed directly.

$$X_{O_2} = X_{PC_2} + R_2 \sin Az_{PC_2O_2} = 5804.344 + 508.2163 \sin (213^\circ 31' 06'') = 5523.7050$$

$$Y_{O_2} = Y_{PC_2} + R_2 \cos Az_{PC_2O_2} = 4911.8798 + 508.2163 \cos (213^\circ 31' 06'') = 4488.1744$$

The distance between the centers of both horizontal curves is found using the Pythagorean theorem while the azimuth is determined using the arctangent function.

$$\begin{aligned} D_{O_1O_2} &= \sqrt{(X_{O_2} - X_{O_1})^2 + (Y_{O_2} - Y_{O_1})^2} \\ &= \left[(5523.7050 - 5294.8021)^2 + (4488.1744 - 4848.6040)^2 \right]^{1/2} \\ &= 426.9731 \end{aligned}$$

$$Az_{O_1O_2} = \tan^{-1} \left[\frac{X_{O_2} - X_{O_1}}{Y_{O_2} - Y_{O_1}} \right] = \tan^{-1} \left[\frac{5523.7050 - 5294.8021}{4488.1744 - 4848.6040} \right] = 147^\circ 34' 51.7''$$

The angle, α , measured at the center of curve #1 between the center of curve #2 and the point of intersection P is calculated using the cosine law.

$$\begin{aligned} \alpha &= \cos^{-1} \left[\frac{D_{O_1O_2}^2 + R_2^2 - R_1^2}{2D_{O_1O_2}R_2} \right] = \cos^{-1} \left[\frac{(426.9731)^2 + (508.2163)^2 - (346.1969)^2}{2(426.9731)(508.2163)} \right] \\ &= 42^\circ 20' 59.6'' \end{aligned}$$

In a similar fashion, the angles β and γ can also be computed with the result that all of the angles being known within the triangle O_1O_2P .

$$\beta = \cos^{-1} \left[\frac{R_1^2 + R_2^2 - D_{O_1O_2}^2}{2R_1D_{O_1O_2}} \right] = \cos^{-1} \left[\frac{(346.1969)^2 + (508.2163)^2 - (426.9731)^2}{2(346.1969)(508.2163)} \right]$$

$$= 56^\circ 11' 05.3''$$

$$\gamma = \cos^{-1} \left[\frac{R_1^2 + D_{O_1O_2}^2 - R_2^2}{2R_1D_{O_1O_2}} \right] = \cos^{-1} \left[\frac{(346.1969)^2 + (426.9731)^2 - (508.2163)^2}{2(346.1969)(426.9731)} \right]$$

$$= 81^\circ 27' 55.1''$$

For a check: $\alpha + \beta + \gamma = 180^\circ 00' 00''$. To compute the coordinates of the point of intersection of the two curves, first compute the azimuth from the center of circle #1 to point P and then use that azimuth to determine the coordinates.

$$Az_{O_1P} = Az_{O_1O_2} - \gamma = (147^\circ 34' 51.7'') - (81^\circ 27' 55.1'') = 66^\circ 06' 56.6''$$

$$X_P = X_{O_1} + R_1 \sin Az_{O_1P} = 5294.8021 + 346.1969 \sin(66^\circ 06' 56.6'') = \underline{5611.352}$$

$$Y_P = Y_{O_1} + R_1 \cos Az_{O_1P} = 4848.6040 + 346.1969 \cos(66^\circ 06' 56.6'') = \underline{4988.776}$$

As a check, compute the coordinates of point P from the center of circle #2.

$$Az_{O_2P} = Az_{O_2O_1} + \alpha = (327^\circ 34' 51.7'') + (42^\circ 20' 59.6'') = 9^\circ 55' 51.3''$$

$$X_P = X_{O_2} + R_2 \sin Az_{O_2P} = 5523.7050 + 508.2163 \sin(9^\circ 55' 51.3'') = \underline{5611.352} \quad (\text{check})$$

$$Y_P = Y_{O_2} + R_2 \cos Az_{O_2P} = 4488.1744 + 508.2163 \cos(9^\circ 55' 51.3'') = \underline{4988.776} \quad (\text{check})$$

The central angle from the PC of each horizontal curve to the point of intersection is calculated next.

$$Az_{O_1PC_1} = Az_{O_1PT_1} + \Delta = (45^\circ 34' 21'') + (26^\circ 15' 33'') = 71^\circ 49' 54''$$

$$\delta_{PC_1P} = Az_{O_1PC_1} - Az_{O_1P} = (71^\circ 49' 54'') - (66^\circ 06' 56.6'') = 5^\circ 42' 57''$$

$$\delta_{PC_2P} = Az_{O_2PC_2} - Az_{O_2P} = (33^\circ 31' 06'') - (9^\circ 55' 51.3'') = 23^\circ 35' 15''$$

The arc length from the PC to P can then be found.

$$c'_{PC_1P} = R_1 \delta_{PC_1P} = (346.1969) \left[\left(5^\circ 42' 57.4'' \right) \frac{\pi}{180^\circ} \right] = 34.537$$

$$c'_{PC_2P} = R_2 \delta_{PC_2P} = (508.2163) \left[\left(23^\circ 35' 14.7'' \right) \frac{\pi}{180^\circ} \right] = 209.222$$

Finally, compute the coordinates of the PC for curve #1. With this known, then the azimuths and chord distances from the two PCs can be simply computed using the arctangent function and the Pythagorean theorem.

$$X_{PC_1} = X_{O_1} + R_1 \sin Az_{O_1PC_1} = 5294.8021 + 346.1969 \sin (71^\circ 49' 54'') = 5623.739$$

$$Y_{PC_1} = Y_{O_1} + R_1 \cos Az_{O_1PC_1} = 4848.6040 + 346.1969 \cos (71^\circ 49' 54'') = 4956.552$$

With the coordinates of both PCs and the point of intersection known, compute the azimuths and distances to P from each PC.

$$Az_{PC_1P} = \tan^{-1} \left[\frac{X_P - X_{PC_1}}{Y_P - Y_{PC_1}} \right] = \tan^{-1} \left[\frac{5611.352 - 5623.739}{4988.776 - 4956.552} \right] = 338^\circ 58' 23''$$

$$D_{PC_1P} = \sqrt{(X_P - X_{PC_1})^2 + (Y_P - Y_{PC_1})^2} = \left[(5611.352 - 5623.739)^2 + (4988.776 - 4956.552)^2 \right]^{1/2} \\ = 34.523$$

$$Az_{PC_2P} = \tan^{-1} \left[\frac{X_P - X_{PC_2}}{Y_P - Y_{PC_2}} \right] = \tan^{-1} \left[\frac{5611.352 - 5804.344}{4988.776 - 4911.879} \right] = 291^\circ 43' 29''$$

$$D_{PC_2P} = \sqrt{(X_P - X_{PC_2})^2 + (Y_P - Y_{PC_2})^2} = \left[(5611.352 - 5804.344)^2 + (4988.776 - 4911.879)^2 \right]^{1/2} \\ = 207.748$$