7

AREA PARTITIONING

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INTRODUCTION

The division of a parcel into two or more separate polygons is a common surveying application. For a rectangular figure, the process is pretty straightforward. Unfortunately, the process becomes a little more complex for irregular figures. This paper will look at the problem from a number of view points. First, the problem of dividing a quadrilateral will be discussed. Then, a figure with more than four sides will be presented and finally, how to divide a polygons with a curved side will be shown

Division is usually performed by (Anderson and Mikhail, 1998; Moffitt and Bossler, 1998):

- Dividing the parcel with a line whose direction is known, such as parallel to another line within the traverse.
- Dividing the parcel with a line from a given starting points on the perimeter of the polygon

SUBDIVIDING A QUADRILATERAL

Stoughton (1986) presents an excellent overview of subdividing a quadrilateral. He has identified six different cases on how the subdivision can be computed. Figure 1 shows the general polygon that will be used to develop the theory Stoughton presented in his paper. The original parcel is defined by points A, B, C, and D whereas the ends of the dividing line are represented by points E and F.

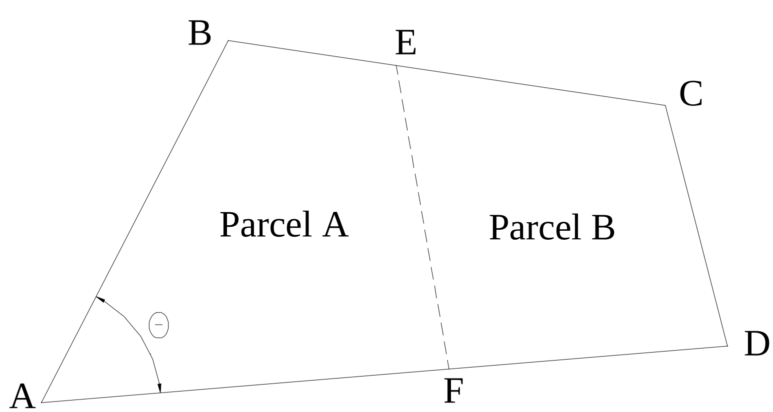

Figure 1. General form of quadrilateral used in developing area partitioning theory.

Case I is the simplest case. Here, lines BC and AD are parallel to each other and the dividing line EF is parallel to one of the sides, such as AB. The perpendicular distance between lines BC and AD, h, is found as

$$h = D_{AB} \sin\theta$$

where θ is the acute angle at point A. The area (A) of the resulting parallelogram that forms parcel A is found by

$$\begin{aligned} A &= D_{FA} h \\ &= D_{FA} D_{AB} \sin\theta \end{aligned}$$

Solving for the unknown distance to the dividing line, D_{FA}

$$D_{FA} = \frac{A}{D_{AB} \sin\theta}$$

Example: Divide the area for the quadrilateral shown in figure 2 into two equal areas with a dividing line parallel to line AB

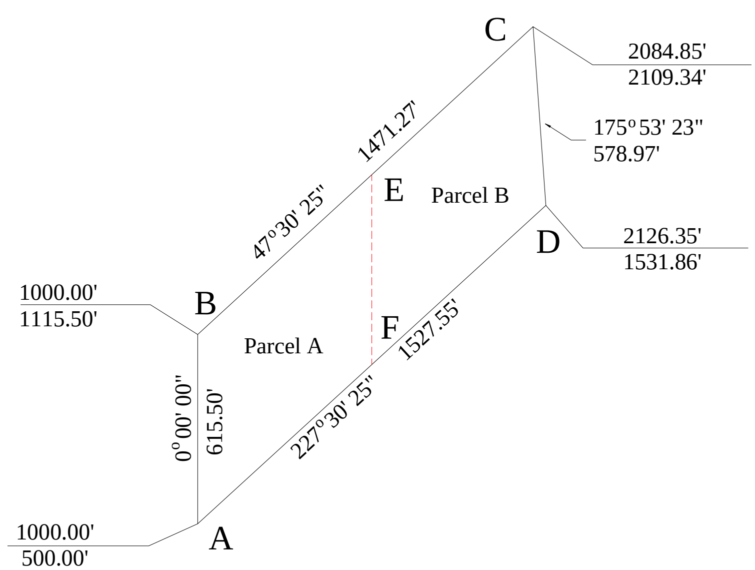

Figure 2. Example quadrilateral with lines AD parallel to BC and dividing line parallel to line AB.

Solution: The following table shows that the total area of polygon ABCD is 680,490 square feet. The desired area for parcel A is therefore 340,245 square feet.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	0	0	0	615.50	0.00	615.50	0.00	0
BC	47	30	25	1471.27	1084.85	993.84	1084.85	1078176.263
CD	175	53	25	578.97	41.49	-577.48	2211.20	-1276927.626
DA	227	30	25	1527.55	-1126.35	-1031.86	1126.34	-1162228.369
					-0.01	0.00	2A =	-1360979.732
							A =	680489.8662

The angle θ is found as the difference in azimuth between lines AD and AB.

$$\theta = 47^{o}30'25'' - 0^{o}00'00'' = 47^{o}30'25''$$

The distance D_{AF} is found using the relationship developed by Stoughton.

$$D_{AF} = \frac{340{,}245 \text{ sq ft}}{615.50' \sin 47^{o}30'25''} = 749.695'$$

Inserting this value for the distance D_{BE} and recognizing that $D_{EF} = D_{AB}$ and $Az_{EF} = Az_{AB} + 180^{o}$, we can see from the results in the following table that the correct distance was found. The slight discrepancy is due to round off.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	0	0	0	615.50	0.00	615.50	0.00	0
BE	47	30	25	749.70	552.79	506.42	552.79	279945.9802
EF	180	0	0	615.50	0.00	-615.50	1105.59	-680490.0503
FA	227	30	25	749.70	-552.79	-506.42	552.79	-279945.9802
					0.00	0.00	2A =	-680490.0503
							A =	340245.0251

While the approach by Stoughton is very simple, the calculations can also be done directly using coordinates. To develop the formula for computing the distance to the dividing line, write four equations for determining the coordinates of the two ends of the line.

$$X_E = X_B + D_{BE} \sin Az_{BE} = X_B + D \sin \alpha$$
$$Y_E = Y_B + D_{BE} \sin Az_{BE} = Y_B + D \cos \alpha$$
$$X_F = X_A + D_{AF} \sin Az_{AF} = X_A + D \sin \alpha$$
$$Y_F = Y_A + D_{AF} \sin Az_{AF} = Y_A + D \cos \alpha$$

In these equations, D is the distance to the dividing line. We simplified the equations taking into account that $D_{AF} = D_{BE} = D$. In addition, since the line BC is parallel to line AD, then $Az_{BE} = Az_{AF} = \alpha$. A fifth equation can also be developed and this is the double area formula for parcel A.

$$2A = X_A\left(Y_B - Y_F\right) + X_B\left(Y_E - Y_A\right) + X_E\left(Y_F - Y_B\right) + X_F\left(Y_A - Y_E\right)$$

Substitute the value for the coordinates into the area equation

$$\begin{aligned} 2A = {} & X_A\left(Y_B - Y_A - D\cos\alpha\right) + X_B\left(Y_B + D\cos\alpha - Y_B\right) \\ & + \left(X_B + D\sin\alpha\right)\left(Y_A + D\cos\alpha - Y_B\right) + \left(X_A + D\sin\alpha\right)\left(Y_A - Y_B - D\cos\alpha\right) \\ = {} & 2\left[\left(X_B - X_A\right)\cos\alpha + \left(Y_A - Y_B\right)\sin\alpha\right]D \end{aligned}$$

Solving for D_{AF} (recall that $D_{AF} = D$ and $Az_{FA} = \alpha$)

$$D_{AF} = \frac{A}{\left(X_B - X_A\right)\cos\alpha + \left(Y_A - Y_B\right)\sin\alpha}$$

Using the data in the example,

$$\begin{aligned} D_{AF} &= \frac{340{,}245}{\left(1000.00 - 1000.00\right)\cos\left(47^\circ 30'25''\right) + \left(500.00 - 1115.50\right)\sin\left(47^\circ 30'25''\right)} \\ &= 749.69' \end{aligned}$$

This corresponds to the same value as shown in the previous example[1].

Case II is very similar to Case I in that line BC is parallel to line AD but here the dividing line is fixed on one of the parallel lines, such as point F along line AD. The result is a trapezoid. The area of a trapezoid is the average length times the height. The height is the perpendicular distance between the two parallel lines. It can be computed as

$$h = D_{AB}\sin\theta$$

The area of parcel A is then defined as

$$A = \frac{D_{AF} + D_{BE}}{2}h$$

Solving for the unknown distance D_{BE} gives us

[1] Note that the value for the distance D_{AF} does come out as a negative number in this calculation. Computing area in a clockwise manner, as done in this example, results in a negative quantity. To alleviate this problem, the area could be entered as a negative value.

$$D_{BE} = \frac{2A}{h} - D_{AF}$$

If this relationship results in a negative number then this means that parcel A is a triangle. The unknown in this case is the altitude h, which can be computed using the relationship

$$h = \frac{2A}{D_{AF}}$$

The distance of the dividing line of the triangle, D_{EF}, can be found from

$$D_{EF} = \frac{2A}{D_{AF} \sin\theta}$$

Using the previous example, divide the polygon ABCD into equal areas where point F is fixed at the midpoint along line AD. To solve this problem, first find the coordinates of point F by averaging the coordinates of the two ends of the line. Thus,

$$X_F = 1{,}563.175'$$
$$Y_F = 1{,}015.930'$$

The perpendicular distance between the two parallel lines is

$$\begin{aligned} h &= D_{AB} \sin\theta = 615.50' \sin\left(47^\circ 30'25''\right) \\ &= 453.845' \end{aligned}$$

The unknown distance in parcel A is next to be calculated

$$\begin{aligned} D_{BE} &= \frac{2A}{h} - D_{AF} = \frac{2\left(340{,}245 \text{ sq ft}\right)}{453.845'} - 763.775' \\ &= 735.614' \end{aligned}$$

The coordinates of point E can now be computed.

$$\begin{aligned} X_E &= X_B + D_{BE} \sin Az_{BE} = 1000.00' + 735.614' \sin 47^\circ 30'25'' \\ &= 1{,}542.41' \end{aligned}$$

$$\begin{aligned} Y_E &= Y_B + D_{BE} \cos Az_{BE} = 1115.50' + 735.614' \cos 47^\circ 30'25'' \\ &= 1{,}612.41' \end{aligned}$$

Using the distance formula and the azimuth relationship, D_{EF} and α_{EF} can be computed.

$$D_{EF} = \sqrt{(1563.175' - 1542.41')^2 + (1015.93' - 1612.41')}$$
$$= 596.84'$$

$$\alpha_{EF} = \tan^{-1}\left[\frac{1563.175' - 1542.41'}{1015.93' - 1612.41'}\right]$$
$$= 178^{o}\ 00'\ 23''$$

Inserting these values into the following spreadsheet shows that the calculations are correct. Again, the slight discrepancy is caused by round-off errors.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	0	0	0	615.50	0.00	615.50	0.00	0
BE	47	30	25	735.61	542.41	496.91	542.41	269528.6767
EF	178	0	23	596.84	20.76	-596.48	1105.59	-659458.7994
FA	227	30	25	763.78	-563.18	-515.93	563.17	-290558.1105
					0.00	0.00	2A =	-680488.2332
							A =	340244.1166

As it was shown in Case I, the coordinates of the unknown point E can be found directly using coordinates of the quadrilateral. Write two equations, one for the double area and one for the azimuth of line BF. Thus, we have

$$2A = X_A(Y_B - Y_F) + X_B(Y_E - Y_A) + X_E(Y_F - Y_B) + X_F(Y_A - Y_E)$$

$$\tan Az_{BE} = \frac{X_E - X_B}{Y_E - Y_B}$$

In the azimuth equation, rewrite it in terms of Y_E.

$$Y_E = (X_E - X_B)\cot Az_{BE} + Y_B$$

Substitute this into the double area equation and solve for X_E. The result is

$$X_E = \frac{2A - X_A(Y_B - Y_F) - Y_A(X_F - X_B) + X_B(X_B - X_F)\cot Az_{BE} + Y_B(X_F - X_B)}{(X_B - X_F)\cot Az_{BE} + Y_F - Y_B}$$

The results of this approach are shown in the following Mathcad program. Note that the area is negative because the area computation is clockwise. Also, the variable $c\alpha$ is the cotangent of the azimuth between B and E.

Area Partitioning of a Quadrilateral Using Coordinates

$$dd(ang) := \begin{vmatrix} degree \leftarrow floor(ang) \\ mins \leftarrow (ang - degree)\cdot 100.0 \\ minutes \leftarrow floor(mins) \\ seconds \leftarrow (mins - minutes)\cdot 100.0 \\ degree + \dfrac{minutes}{60.0} + \dfrac{seconds}{3600.0} \end{vmatrix}$$

$$radians(ang) := \begin{vmatrix} d \leftarrow dd(ang) \\ d\cdot\dfrac{\pi}{180.0} \end{vmatrix}$$

The given quantities are:

$A := -340245$

$X_A := 1000.00$ $\quad Y_A := 500.00$ $\quad Az_{BC} := 47.3025$

$X_B := 1000.00$ $\quad Y_B := 1115.50$ $\quad c\alpha := \dfrac{1}{\tan\left(radians\left(Az_{BC}\right)\right)}$

$X_D := 2126.35$ $\quad Y_D := 1531.86$ $\quad c\alpha = 0.9161082$

The mid-point along line AD is found using the average coordinates of the ends of the line

$$X_F := \frac{X_A + X_D}{2} \qquad Y_F := \frac{Y_A + Y_D}{2}$$

Solving for the coordinates of the unknown point E:

$$X_E := \frac{\left[2\cdot A - X_A\cdot\left(Y_B - Y_F\right)\right] - Y_A\cdot\left(X_F - X_B\right) + X_B\cdot\left(X_B - X_F\right)\cdot c\alpha + Y_B\cdot\left(X_F - X_B\right)}{\left(X_B - X_F\right)\cdot c\alpha + Y_F - Y_B}$$

$$X_E = 1542.415$$

$$Y_E := \left(X_E - X_B\right)\cdot c\alpha + Y_B \qquad Y_E = 1612.410$$

Because the quadrilateral is such a simple figure, the approach used by Stoughton is easier to solve, unless the solution is already encoded in a program. The remaining cases presented by Stoughton (1986) will only be discussed using the geometric method he uses in the paper. Later on, a generalized approach using coordinates for polygons of any size will be presented.

Case III approaches the problem a little differently. Here lines BC and AD are not parallel. The dividing line, though, is parallel to line AB. Stoughton (1986) identifies four solutions based on the location of the acute angles. These solutions are:

a. Case IIIa, $\angle_A$ and $\angle_B$ are acute,
b. Case IIIb, $\angle_A$ and $\angle_E$ are acute,
c. Case IIIc, $\angle_B$ and $\angle_F$ are acute, and
d. Case IIId, $\angle_E$ and $\angle_F$ are acute.

Case IIIa is depicted in figure 3. The distance D_{AB} is made up of three segments, namely u + v + w, where v is equal to the distance between E and F (D_{EF}). The acute angles are θ_1 and θ_2 located at A and B respectively. The distance h is the perpendicular distance between line AB and the dividing line EF. Using trigonometry, these angels are shown as:

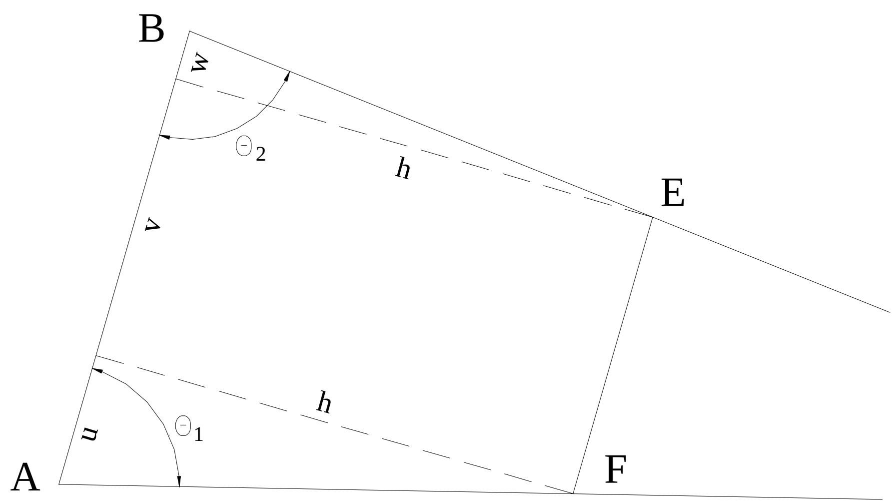

Figure 3. The geometry of Case IIIa, from Stoughton (1986).

$$\tan\theta_1 = \frac{h}{u} \quad \Rightarrow \quad u = h\cot\theta_1$$
$$\tan\theta_2 = \frac{h}{w} \quad \Rightarrow \quad w = h\cot\theta_2$$

The distance from A to B can be represented as

$$D_{AB} = h\cot\theta_1 + D_{EF} + h\cot\theta_2$$

The area of the trapezoid ABEF is

$$\begin{aligned} A &= \frac{h}{2}\left(D_{AB} + D_{EF}\right) \\ &= D_{AB}h - \frac{\cot\theta_1 + \cot\theta_2}{2}h^2 \end{aligned}$$

This can be shown in the following form.

$$\left(\frac{\cot\theta_1 + \cot\theta_2}{2}\right)h^2 - D_{AB}\,h + A = 0$$

The solution is performed using the quadratic equation.

$$h = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where: $a = \dfrac{\cot\theta_1 + \cot\theta_2}{2}$

$b = -D_{AB}$

$c = A$

Example: The problem is to find the location of the dividing line, EF, within polygon ABCD shown in figure 4 such that the area of both new parcels are equal. From analysis of the data, the desired area for parcel A is 316,212.234 square feet.

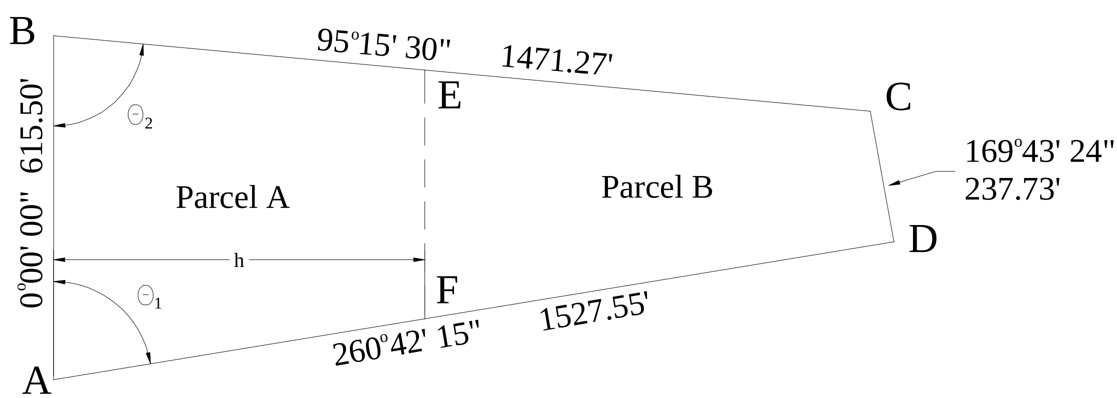

Figure 4. Example using Case IIIa from Stoughton (1986).

Solution: The acute angles are computed first.

$$\theta_1 = (80^\circ 42'15'') - (0^\circ 00'00'') = 80^\circ\ 42'15''$$

$$\theta_2 = (180^\circ 00'00'') - (95^\circ 15'30'') = 84^\circ\ 44'30''$$

Then,

$$a = \frac{\cot\left(80^{o}\,42'15''\right) + \cot\left(84^{o}\,44'30''\right)}{2} = 0.127858$$

$$b = -615.50'$$

$$c = 316{,}212.234\ \text{ft}^{2}$$

The offset distance, h, is found using the quadratic equation.

$$h = \frac{-\left(-615.50'\right) \pm \sqrt{\left(-615.50'\right)^{2} - 4\left(0.127858\right)\left(316212.234\ \text{ft}^{2}\right)}}{2\left(0.127858\right)}$$

$$= 4229.16' \quad \text{or} \quad 584.787'$$

The obvious correct answer is 584.787'. The distances D_{AF} and D_{BE} are found to be

$$D_{AF} = \frac{h}{\sin\theta_1} = \frac{584.787'}{\sin 80^{o}\,42'15''} = 592.57'$$

$$D_{BE} = \frac{h}{\sin\theta_2} = \frac{584.787'}{\sin 84^{o}\,44'30''} = 587.26'$$

The distance of the dividing line becomes

$$D_{EF} = 615.50' - 584.787'\cot\left(80^{o}\,42'15''\right) - 584.787'\cot\left(84^{o}\,44'30''\right)$$
$$= 465.96'$$

As shown in the following spreadsheet used to compute the area of parcel A, the answer passes the check.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	0	0	0	615.50	0.00	615.50	0.00	0
BE	95	15	30	587.26	584.79	-53.82	584.79	-31473.49401
EF	180	0	0	465.96	0.00	-465.96	1169.58	-544976.1736
FA	260	42	15	592.57	-584.79	-95.72	584.79	-55975.4574
					0.00	0.00	2A =	-632425.125
							A =	316212.5625

In Case IIIb, the acute angles are located at points A and E (figure 5). The dividing line, EF, is parallel to line AB. The angles θ_1 and θ_2 can be defined as

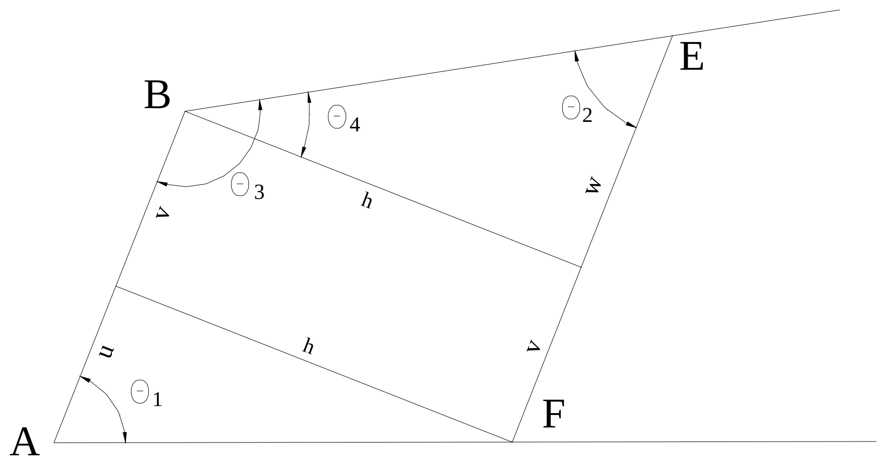

Figure 5. Geometry for subdivision of a quadrilateral using Case IIIb (from Stoughton, 1986).

$$\tan\theta_1 = \frac{h}{u} \quad \Rightarrow \quad u = h\cot\theta_1$$

$$\tan\theta_2 = \frac{h}{w} \quad \Rightarrow \quad w = h\cot\theta_2$$

Again, the area of the trapezoid ABEF is

$$\begin{aligned} A &= \frac{h}{2}\left(D_{AB} + D_{EF}\right) \\ &= \frac{\cot\theta_2 - \cot\theta_1}{2}\, h^2 + D_{AB} h \end{aligned}$$

Therefore,

$$\frac{\cot\theta_2 - \cot\theta_1}{2}\, h^2 + D_{AB} h - A = 0$$

Solve the equation using the quadratic equation.

Example: Given the data shown in figure 6, find the location of the dividing line that divides the area in half, where the dividing line is parallel to line AB.

Solution: The total area of the quadrilateral is 648,421.1502 square feet. Half of this area is 324,210.58 square feet.

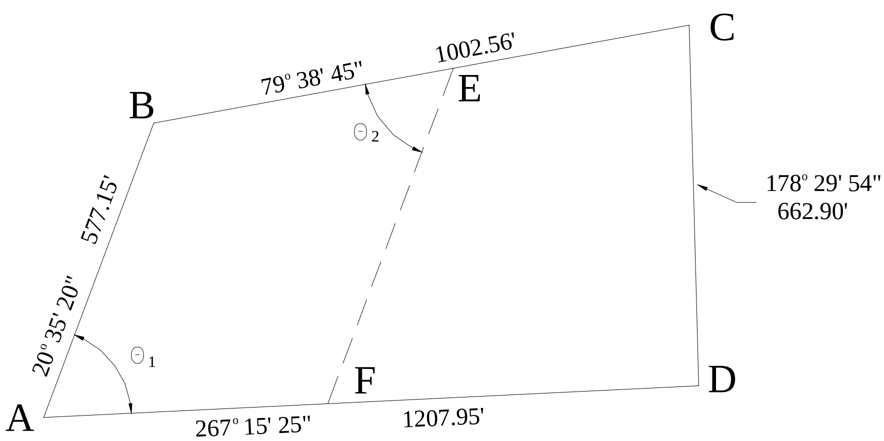

Figure 6. Example area partitioning using Case IIIb.

Then,

$$\theta_1 = \left(87^\circ 15'25''\right) - \left(20^\circ 35'20''\right) = 66^\circ\ 40'\ 05''$$

$$\theta_2 = \left(79^\circ 38'45''\right) - \left(20^\circ 35'20''\right) = 59^\circ\ 03'\ 25''$$

$$a = \frac{\cot\left(59^\circ 03'25''\right) - \cot\left(66^\circ 40'05''\right)}{2} = 0.8408980$$

$$b = 577.15'$$

$$c = -324{,}210.58 \text{ sq. ft.}$$

The offset distance of the dividing line from line AB is

$$h = \frac{-577.15' \pm \sqrt{\left(577.15\right)^2 - 4\left(0.8408980\right)\left(-324{,}210.58\right)}}{2\left(0.8408980\right)}$$

$$= 522.038' \quad \text{and} \quad -7{,}385.53'$$

The obvious correct answer is 522.038’. The distances D_{BE} and D_{AF} become

$$D_{BE} = \frac{522.038'}{\sin 59^{\circ}03'25''} = 608.66'$$

$$D_{AF} = \frac{522.038'}{\sin 66^{\circ}50'05''} = 568.53'$$

The line segments u and w are also found using basic trigonometric relationships.

$$u = \frac{522.038'}{\tan 66^{\circ}40'05''} = 225.170'$$

$$w = \frac{522.038'}{\tan 59^{\circ}03'25''} = 312.966'$$

The length of dividing line EF becomes

$$D_{EF} = 577.15' - 225.170' + 312.966'$$
$$= 664.946'$$

As a check, compute the area of quadrilateral ABEF. The results are shown in the next table and it can be seen that the dividing line has been correctly located.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	20	35	20	577.15	202.96	540.29	202.96	109656.8166
BE	79	38	45	608.66	598.75	109.40	1004.67	109906.687
EF	200	35	20	664.95	-233.84	-622.48	1369.58	-852534.5569
FA	267	15	25	568.53	-567.88	-27.21	567.87	-15450.61308
					-0.01	0.00	2A =	-648421.6663
							A =	324210.8332

While Stoughton uses the acute angle at E, one can also use the angle at point B. The acute angle at B opposite the line segment w can be written by the following trigonometric relationship (figure 5).

$$\tan\theta_4 = \frac{w}{h} \quad \Rightarrow w = h\tan\theta_4$$

Since $\theta_4 = \theta_3 - 90^{o}$, one can use θ_3 directly using the trigonometric identity

$$\tan\left(\theta - 90^{o}\right) = -\cot\theta$$

Therefore, the line segment w becomes

$$w = -h \cot \theta_3$$

Then, the area function can be represented as

$$\frac{-\cot \theta_1 - \cot \theta_3}{2} h^2 + D_{AB} h - A = 0$$

From the example, a can be found to be

$$a = \frac{-\cot\left(66^\circ 40'05''\right) - \cot\left(120^\circ 56'35''\right)}{2} = 0.8408980$$

which is identical to the value using the acute angle at E. Therefore, the solution will be the same.

Stoughton (1986) identifies Case IIIc where lines BC and AD are not parallel but the dividing line EF is parallel to line AB. The acute angles (θ_1 and θ_2) are located at points B and F (figure 7). The line segments u and w are found as before.

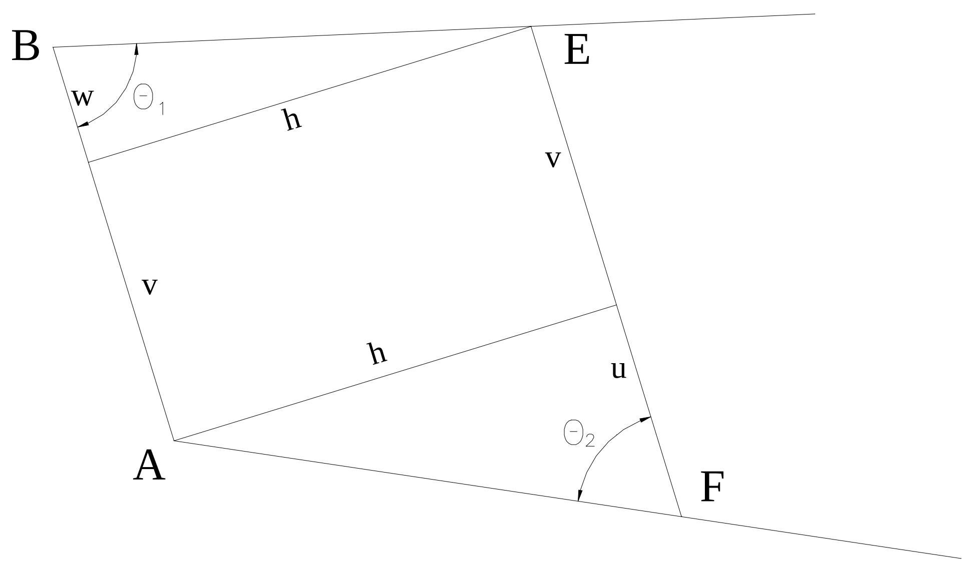

Figure 7. Geometry for Case IIIc after Stoughton (1986).

$$u = h \cot \theta_1$$

$$w = h \cot \theta_2$$

Substitute these values into the area of the quadrilateral.

$$A = \frac{h}{2}\left(D_{AB} + D_{EF}\right)$$

$$\frac{\cot\theta_1 - \cot\theta_2}{2} h^2 + D_{AB}h - A = 0$$

Example. Given the data in figure 8, find the location of the dividing line EF that is parallel to line AB such that both of the new parcels have the same area.

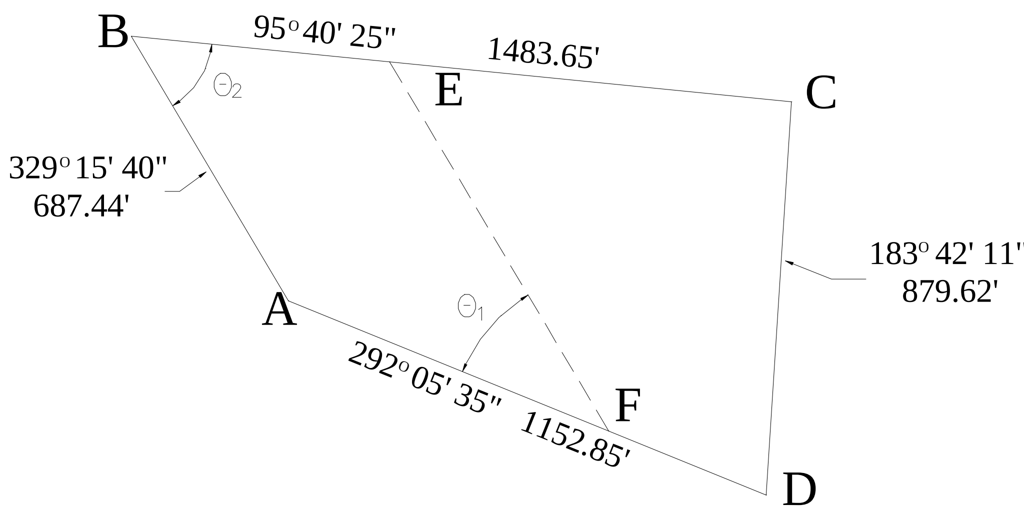

Figure 8. Example for Stoughton's Case IIIc.

Solution. The total area of the quadrilateral is 891,539.35 square feet. The desired area for the two parcels is 445,769.61 square feet. The acute angles are

$$\theta_1 = \left(329^\circ 15'40''\right) - \left(292^\circ 05'35''\right) = 37^\circ\ 10'\ 05''$$

$$\theta_2 = \left(149^\circ 15'40''\right) - \left(95^\circ 40'25''\right) = 53^\circ\ 35'15''$$

$$a = \frac{\cot\left(37^\circ 10'05''\right) - \cot\left(53^\circ 35'15''\right)}{2} = 0.2906887$$

$$b = 687.44'$$

$$c = -445{,}769.677 \text{ sq ft}$$

$$h = \frac{-687.44' \pm \sqrt{(687.44')^2 - 4(0.2906887)(-445{,}769.677 \text{ sq ft})}}{2(0.2906887)}$$

$$= 529.771' \quad \text{or} \quad -2{,}894.64'$$

The obvious correct answer is 529.771'. The distances of the dividing line and the offsets from line AB to line EF are

$$\begin{aligned} D_{EF} &= D_{AB} - h\cot\theta_2 + h\cot\theta_1 \\ &= 687.44' - 529.771'\left[\cot\left(37^\circ 10'05''\right) - \cot\left(53^\circ 35'15''\right)\right] \\ &= 995.44' \end{aligned}$$

$$D_{BE} = \frac{529.771'}{\sin 53^\circ 35'15''} = 658.293'$$

$$D_{FA} = \frac{529.771'}{\sin 37^\circ 10'05''} = 876.879'$$

A check of the results are shown in the following table. The slight discrepancy is due to round off errors.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	329	15	40	687.44	-351.37	590.86	-351.37	-207609.2064
BE	95	40	25	658.29	655.07	-65.08	-47.67	3102.31006
EF	149	15	40	995.44	508.80	-855.59	1116.19	-955000.6245
FA	292	5	35	876.88	-812.49	329.80	812.50	267965.2193
					0.00	0.00	2A =	-891542.3016
							A =	445771.1508

Stoughton identifies Case IIId by the situation where the acute angles occur at the dividing line, namely points E and F (figure 9). The distance D_{EF} is (recognizing that v = D_{AB})

$$D_{EF} = u + D_{AB} + w$$

The line segments are

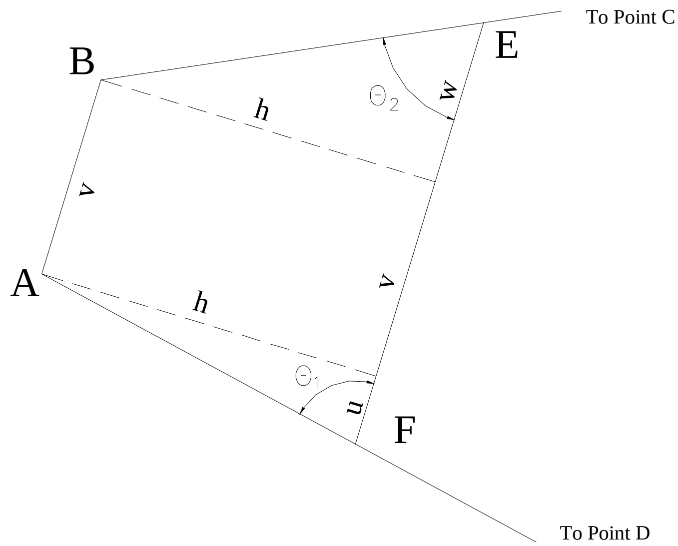

Figure 9. Geometry for the area partitioning using Case IIId from Stoughton (1986).

$$\tan\theta_1 = \frac{h}{u} \quad \Rightarrow \quad u = h\cot\theta_1$$

$$\tan\theta_2 = \frac{h}{w} \quad \Rightarrow \quad w = h\cot\theta_2$$

Therefore,

$$D_{EF} = D_{AB} + h\cot\theta_1 + h\cot\theta_2$$

The area of the quadrilateral is

$$A = \frac{D_{AB} + D_{EF}}{2}h = \frac{h}{2}\left(D_{AB} + D_{AB} + h\cot\theta_1 + h\cot\theta_2\right)$$

from which

$$\frac{\cot\theta_1 + \cot\theta_2}{2}h^2 + D_{AB}h - A = 0$$

Example. Using figure 9, the results of the area computation of quadrilateral ABCD is given in the following table. Divide the area in half with the dividing line oriented parallel to line AB.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	12	15	20	582.65	123.68	569.37	123.68	70420.1949
BC	52	38	5	1216.85	967.13	738.50	1214.49	896901.6812
CD	182	33	1	2391.98	-106.43	-2389.61	2075.19	-4958895.914
DA	317	41	53	1462.59	-984.38	1081.74	984.38	1064844.599
					0.00	0.00	2A =	-2926729.439
							A =	1463364.719

Solution. See the following Mathcad program for the solution to this example. Note that $c\theta_I$ represents the cotangent of θ_i.

Solution to Area Partitioning Problem Using Case IIId as Presented by Stoughton

$$dd(ang) := \left| \begin{array}{l} degree \leftarrow floor(ang) \\ mins \leftarrow (ang - degree)\cdot 100.0 \\ minutes \leftarrow floor(mins) \\ seconds \leftarrow (mins - minutes)\cdot 100.0 \\ degree + \dfrac{minutes}{60.0} + \dfrac{seconds}{3600.0} \end{array} \right.$$

$$radians(ang) := \left| \begin{array}{l} d \leftarrow dd(ang) \\ d\cdot\dfrac{\pi}{180.0} \end{array} \right.$$

The given values for this problem are:

$\theta_1 := 54.3327$

$\theta_2 := 40.2245$

$D_{AB} := 582.65$

$A := -731682.36$

$$c\theta_1 := \frac{1}{\tan(radians(\theta_1))}$$

$$c\theta_2 := \frac{1}{\tan(radians(\theta_2))}$$

The Solution is:

$a := \dfrac{c\theta_1 + c\theta_2}{2}$ $\quad b := D_{AB}$ $\quad c := A$

$$a\cdot h^2 + b\cdot h + c \text{ solve}, h \rightarrow \begin{pmatrix} -1241.6760509586842276 \\ 624.34500948755793002 \end{pmatrix}$$

$h := 624.345$

$u := h \cdot c\theta_1$ $u = 444.396$

$w := h \cdot c\theta_2$ $w = 734.144$

$D_{EF} := D_{AB} + u + w$ $D_{EF} = 1761.19$

$D_{AF} := \dfrac{h}{\sin\left(\text{radians}\left(\theta_1\right)\right)}$ $D_{AF} = 766.352$

$D_{BE} := \dfrac{h}{\sin\left(\text{radians}\left(\theta_2\right)\right)}$ $D_{BE} = 963.729$

Check the results. They are given for parcel ABEF in the next table.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	12	15	20	582.65	123.68	569.37	123.68	70420.1949
BE	52	38	5	963.73	765.95	584.88	1013.32	592669.9522
EF	192	15	20	1761.19	-373.85	-1721.05	1405.42	-2418801.125
FA	317	41	53	766.35	-515.78	566.80	515.78	292346.0441
					0.00	0.00	2A =	-1463364.933
							A =	731682.4667

A rather interesting geometric approach to solving the Case III area partitioning problem was presented by Stoughton (1986). This involved projecting the lines BC and AD to a point of intersection (point P in figures 10 and 11). There are two possible scenarios for this geometric construction. The first is when the dividing line lies between the line AB and the point of intersection as depicted in figure 10. The other is when the line AB lies between the dividing line and the point of intersection (figure 11).

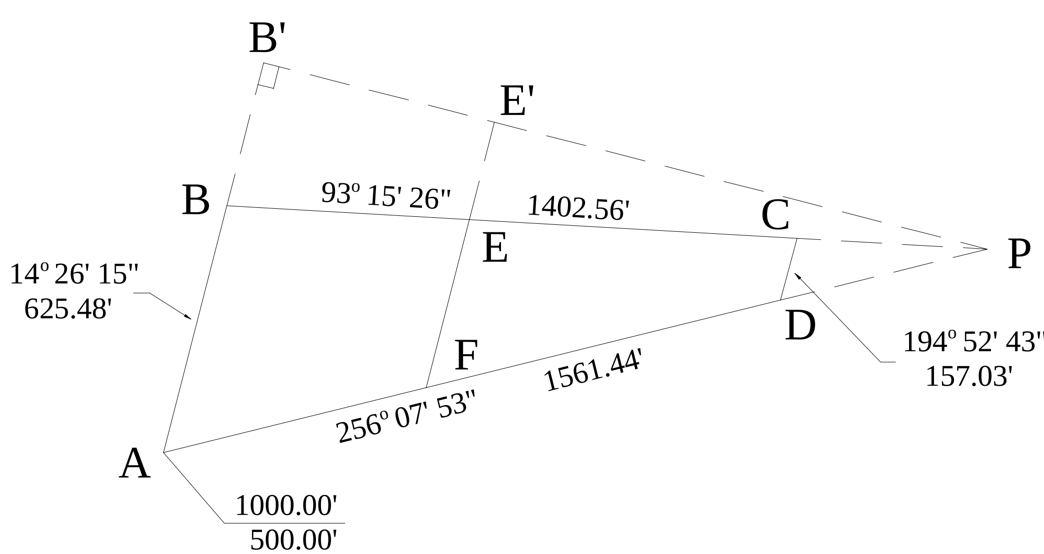

Figure 10. General solution to the Case III area partitioning problem where the dividing line lies between line AB and point P.

Lets look at the first configuration where the dividing line is between line AB and point P. The height or altitude of triangle ABP is designated as h and defined by the line B'P. The coordinates of point P can be determined using a line-line intersection algorithm. Then, using the sine law, the angle at A can be represented as

$$\sin\angle_A = \frac{h}{D_{AP}}$$

Defining $\Delta_X = X_P - X_A$ and $\Delta_Y = Y_P - Y_A$, then $\sin\left(Az_{DA} + 180^o\right) = -\sin Az_{DA}$ and $\cos\left(Az_{DA} + 180^o\right) = -\cos Az_{DA}$. From this one can write

$$\Delta_X = -D_{AP}\sin Az_{AD} \quad\Rightarrow\quad \sin Az_{AD} = -\frac{\Delta_X}{D_{AP}}$$

$$\Delta_Y = -D_{AP}\cos Az_{AD} \quad\Rightarrow\quad \cos Az_{AD} = -\frac{\Delta_Y}{D_{AP}}$$

The height of triangle ABP becomes

$$\begin{aligned} h &= D_{AP}\sin\angle_A \\ &= D_{AP}\sin\left(Az_{AD} - Az_{AB}\right) \\ &= D_{AP}\left(\sin Az_{AD}\cos Az_{AB} - \cos Az_{AD}\sin Az_{AB}\right) \\ &= D_{AP}\left(-\sin Az_{DA}\cos Az_{AB} + \cos Az_{DA}\sin Az_{AB}\right) \end{aligned}$$

Substitute the sines and cosines of line AD

$$\begin{aligned} h &= D_{AP}\left(\frac{\Delta_X\cos Az_{AB}}{D_{AP}} - \frac{\Delta_Y\sin Az_{AB}}{D_{AP}}\right) \\ &= \Delta_X\cos Az_{AB} - \Delta_Y\sin Az_{AB} \end{aligned}$$

The area of triangle ABP is

$$A_{ABP} = \frac{1}{2}D_{AB}h$$

The area of triangle EFP can be defined as

$$A_{EFP} = A_{ABP} - A_{ABEF}$$

Using similar triangles, one can write the following ratios.

$$\frac{D_{EF}}{D_{AB}} = \frac{D_{FP}}{D_{AP}} = \frac{D_{EP}}{D_{BP}} = \frac{h'}{h} = k$$

where h' is the height or triangle EFP shown by the distance E'P in figure 10. We know that the area will increase by the square of the ratio of the lengths. In other words,

$$\frac{A_{EFP}}{A_{ABP}} = k^2$$

Since the two areas are known, the unknown distances to triangle EFP can be computed.

$$D_{EF} = k\,D_{AB}$$

$$D_{FP} = k\,D_{AP}$$

$$D_{EP} = k\,D_{BP}$$

Then,

$$D_{BE} = D_{BP} - D_{EP}$$

$$D_{AF} = D_{AP} - D_{FP}$$

Example. Given the data shown in figure 10, where the area of the quadrilateral ABCD is 537,797.4441 square feet, find the location of the dividing line EF such that the areas of the two new parcels are the same.

Solution. The desired area of quadrilateral ABEF is 268,898.7221 square feet. The solution to the problem is given in the following Mathcad program.

Solution to Area Partitioning Problem Using the General form of Case III as Presented by Stoughton

```
dd(ang) := | degree ← floor(ang)
           | mins ← (ang − degree)·100.0
           | minutes ← floor(mins)
           | seconds ← (mins − minutes)·100.0
           | degree + minutes/60.0 + seconds/3600.0

radians(ang) := | d ← dd(ang)
                | d·π/180.0
```

The given values for this problem are:

$Az_{AB} := 14.2615$

$Az_{AD} := 76.0753$

$Az_{BC} := 93.1526$

$D_{AB} := 625.48$

$D_{BC} := 1402.56$

$D_{CD} := 157.03$

$D_{DA} := 1561.44$

$A_{ABEF} := 268898.7221$

$\alpha_{AB} := \text{radians}(Az_{AB})$

$\alpha_{AD} := \text{radians}(Az_{AD})$

$\alpha_{BC} := \text{radians}(Az_{BC})$

$X_A := 1000.00$ (assumed)

$Y_A := 500.00$ (assumed)

$X_B := X_A + D_{AB} \cdot \sin(\alpha_{AB})$

$Y_B := Y_A + D_{AB} \cdot \cos(\alpha_{AB})$

The solution is as follows:

The angles at the vertices of triangle A, B, P are designated by A_i

First, find the coordintes of point P through line-line intersection.

$A_A := \alpha_{AD} - \alpha_{AB}$ $\qquad A_A = 1.0768$

$A_B := (\alpha_{AB} + \pi) - \alpha_{BC}$ $\qquad A_B = 1.7659$

$A_P := \pi - (A_A + A_B)$ $\qquad A_P = 0.2989$

$D_{BP} := \frac{D_{AB}}{\sin(A_P)} \cdot \sin(A_A)$ $\qquad D_{BP} = 1870.0955$

$D_{AP} := \frac{D_{AB}}{\sin(A_P)} \cdot \sin(A_B)$ $\qquad D_{AP} = 2083.7675$

$X_P := X_B + D_{BP} \cdot \sin(\alpha_{BC})$ $\qquad X_P = 3023.0214$

$Y_P := Y_B + D_{BP} \cdot \cos(\alpha_{BC})$ $\qquad Y_P = 999.4712$

Next, perform the steps for the area partitioning

$\Delta_X := X_P - X_A$ $\qquad \Delta_X = 2023.0214$

$\Delta_Y := Y_P - Y_A$ $\qquad \Delta_Y = 499.4712$

$h := \Delta_X \cdot \cos(\alpha_{AB}) - \Delta_Y \cdot \sin(\alpha_{AB})$ $\qquad h = 1834.6048$

$A_{ABP} := \frac{1}{2} \cdot D_{AB} \cdot h$ $\qquad A_{ABP} = 573754.2912$

$A_{EFP} := A_{ABEF} - A_{ABP}$ $\qquad A_{EFP} = -304855.5691$

$k := \sqrt{\left|\frac{A_{EFP}}{A_{ABP}}\right|}$ $k = 0.7289$

$D_{EF} := k \cdot D_{AB}$ $D_{EF} = 455.9293$

$D_{FP} := k \cdot D_{AP}$ $D_{FP} = 1518.9146$

$D_{EP} := k \cdot D_{BP}$ $D_{EP} = 1363.1633$

$D_{BE} := D_{BP} - D_{EP}$ $D_{BE} = 506.9322$

$D_{AF} := D_{AP} - D_{FP}$ $D_{AF} = 564.8529$

As a check, compute the area of the quadrilateral ABEF. The results are shown in the following table. The slight discrepancy is due to round-off errors.

	Azimuth							
Side	Degrees	Minutes	Seconds	Length	Departure	Latitude	DMD	Double Area
AB	14	26	15	625.48	155.95	605.73	155.95	94461.39913
BE	93	15	26	506.93	506.11	-28.80	818.01	-23561.21029
EF	194	26	15	455.93	-113.67	-441.53	1210.45	-534449.3563
FA	256	7	53	564.85	-548.39	-135.39	548.39	-74247.6845
					0.00	0.00	2A =	-537796.852
							A =	268898.426

Division of Polygon Using an Initial Guess of the Dividing Line

A common approach to partitioning an area of a polygon with more than four sides is by establishing an initial guess of where the dividing line may be located. This position is then adjusted to meet the desired area (Moffitt and Bossler, 1998; Anderson and Mikhail, 1998).

Lets first take a look at the first case where the line is of a known direction. Figure 11 shows an example polygon with 6 sides. The problem is to divide the main polygon into two polygons of equal area using a line that is parallel to line AF. This is depicted as line HI in the figure. The solution can be broken down to four main steps.

1) First, compute the area of the entire polygon.

2) Draw a line from one of the nodes on the polygon. In this example, we will construct a line BG that is parallel to line AF. Compute the area of the new polygon ABGFA.

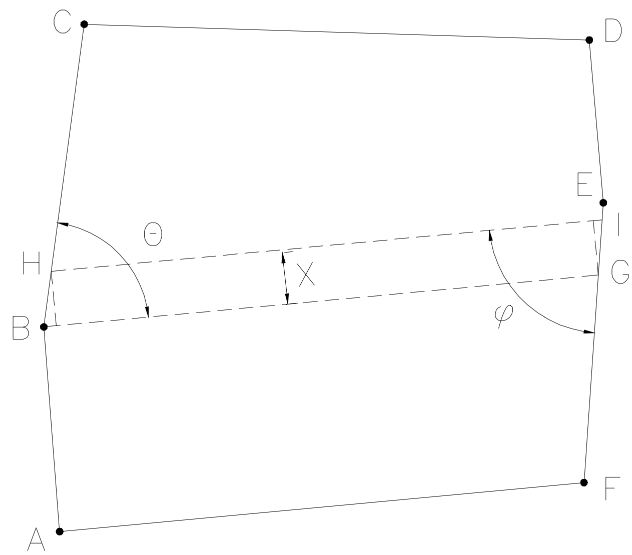

Figure 11. Example area partitioning problem using a line parallel to another line.

3) Chances are that this new polygon will not be equal to half the total area of the original polygon. Assume that the area is too small. Then we will need to add the trapezoid BHIG to the second polygon to arrive at the correct area. To do this we will need to perform a series of calculations to determine the unknowns within the trapezoid.
   a) Compute the distances D_{BG} and D_{GF} along with the angles θ and φ.
   b) Find the distance D_{HI} using the expression

$$D_{HI} = D_{BG} - X\cot\theta + X\cot\varphi$$

   c) Compute the area using the next relationship

$$A_{BHIG} = \frac{1}{2}X\left[D_{BG} + D_{BG} - X\left(\cot\theta - \cot\varphi\right)\right]$$

$$= X\left(D_{BG}\right) - \left(\frac{\cot\theta - \cot\varphi}{2}\right)X^2$$

   d) The solution to the distance offset of line HI to BG is calculated using the quadratic equation.

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where,

$$a = \frac{\cot\theta - \cot\varphi}{2}$$
$$b = -D_{BG}$$
$$C = A_{BHIG}$$

4) Finally, determine the distances D_{BH} and D_{GI}.

$$D_{BH} = \frac{X}{\sin\theta}$$

$$D_{GI} = \frac{X}{\sin\varphi}$$

5) Check the results by computing the area of the two new polygons.

Example (Hashimi, 1988): The following table presents the information pertaining to a 6-sided polygon shown in figure 1:

Side	Azimuth	Distance	Departure	Latitude	DMD	Double Area
AB	340^o 12’	320.4’	-108.5’	+301.5’	108.5	+32,713
BC	5^o 38’	618.6’	+60.7’	+615.6’	60.7	+37.367
CD	96^o 26’	654.3’	+650.2’	-73.3’	771.6	-56,558
DE	174^o 34’	200.6’	+19.0’	-199.7’	1440.8	-287,728
EF	182^o 27’	447.5’	-19.1’	-447.1’	1440.7	-644,137
FA	251^o 53’	633.7’	-602.3’	-197.0’	819.3	-161,402
			Σ = 0.0’	Σ = 0.0’	Double Area = 1,079,745	
					Area = 539,870 sq ft	

The problem is to divide the area into two equal areas using a line that is parallel to line AF in the original polygon. The solution is shown as follows. First we will compute the distance between B and G using a line-line intersection. Then the distance between G and F will be computed by inversing between the coordinates. The following values will be used.

$$x_1 = x_B = -108.5' \qquad x_2 = x_E = +621.4'$$
$$y_1 = y_B = +301.5' \qquad y_2 = y_E = +644.1'$$
$$x_F = +602.3' \qquad y_F = +197.0'$$
$$m_1 = \cot 71^o\ 53' \qquad m_2 = \cot 182^o\ 27'$$

$$x_G = \frac{m_1 x_1 - m_2 x_2 - y_1 + y_2}{m_1 - m_2}$$
$$= \frac{(0.327172)(-108.5) - (23.371777)(621.4) - 301.5 + 644.1}{0.327172 - 23.371777}$$
$$= 616.9'$$

$$y_G = m_1\,(x_G - x_1) + y_1 = 0.327172\,[616.9 - (-108.5)] + 301.5$$
$$= 538.8'$$

$$D_{BG} = \sqrt{(x_G - x_B)^2 + (y_G - y_B)}$$
$$= \sqrt{[616.9 - (-108.5)]^2 + [538.8 - 301.5]^2}$$
$$= 763.2'$$

$$D_{GF} = \sqrt{(x_F - x_G)^2 + (y_F - y_G)^2}$$
$$= \sqrt{(602.3 - 616.9)^2 + (197.0 - 538.8)^2}$$
$$= 342.2'$$

$$Az_{BG} = AZ_{AF}$$

$$\theta = Az_{BG} - Az_{BC} = (71^o 53') - (5^o 38')$$
$$= 66^o\ 15'$$

$$\varphi = Az_{GB} - Az_{GF} = (251^o 53') - (182^o 27')$$
$$= 69^o\ 26'$$

Next, compute the area of the new polygon ABGFA. The results are shown in the following table.

Side	Azimuth	Distance	Departure	Latitude	DMD	Double Area
AB	340^o 12’	320.4’	-108.5’	+301.5’	108.5	+32,713
BG	71^o 53’	763.2’	+725.4’	+237.3’	725.4	+172,137
GF	182^o 27’	342.2’	-14.6’	-341.8’	1436.2	-490,893
FA	251^o 53’	633.7’	-602.3’	-197.0’	819.3	-161,402
			Σ = 0.0’	Σ = 0.0’	Double Area =	447,445
					Area =	223,720 sq ft

Evaluate how close the new polygon is to the desired area.

Total Area = 539,870 sq. ft.
Half Area = 269,935 sq. ft.

A_{ABGF} = $\underline{223{,}720}$ sq. ft.
A_{BGIH} = 46,720 sq. ft $\Rightarrow$ Area to be added to the polygon ABGF

Since the area of the initial division does not divide the original polygon in half, we need to add polygon BGIH to arrive at the correct area. To do this, we need to compute the offset distance X using the quadratic equation.

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where:

$$a = \frac{\cot\theta - \cot\varphi}{2} = \frac{\cot 66^o15' - \cot 69^o26'}{2} = 0.0323995$$

$$b = D_{BG} = 763.2'$$

$$c = A_{BIG} = 46{,}215 \text{ sq. ft.}$$

Then,

$$X = \frac{763.2 - \sqrt{(763.2)^2 - 4(0.0323995)(46{,}215)}}{2(0.0323995)} = 60.7'$$

With this data, compute the distances BH, GI, CH, FI, and HI using the following relationships.

$$D_{BH} = \frac{X}{\sin\theta} = \frac{60.7'}{\sin 66^o15'} = 66.3'$$

$$D_{GI} = \frac{X}{\sin\varphi} = \frac{60.7'}{\sin 69^o26'} = 64.8'$$

$$D_{CH} = D_{BC} - D_{BH} = 618.6' - 66.3' = 552.3'$$

$$D_{FI} = D_{FG} + D_{GI} = 342.2' + 64.8' = 407.0'$$

$$D_{HI} = D_{BA} - X\cot\theta + X\cot\varphi$$
$$= 320.4' - 60.7\cot 66^o15' + 60.7\cot 69^o26' = 759.3'$$

Next, check the results.

	Azimuth						
Side	Degrees	Minutes	Length	Departure	Latitude	DMD	Double Area
AB	340	12.0	320.4	-108.5	301.5	-108.5	-32717.74954
BH	5	38.0	66.3	6.5	66.0	-210.6	-13892.38545
HI	71	53.0	759.3	721.7	236.1	517.6	122211.3333
IF	182	27.0	407.0	-17.4	-406.6	1221.9	-496846.774
FA	251	53.0	633.7	-602.3	-197.1	602.2	-118661.6385
				0.0	-0.1	2A =	-539907.2142
						A =	269953.6071

The slight discrepancy between the results here and the true half area are due to round-off errors in the calculations. Carrying values out to more significant figures would result in a much closer value to the true half area desired. This result is well within the significant figures for the example.

The next example is to divide the polygon using a defined point on the edge of the polygon. For example, figure 12 shows the same polygon except that this time the division will be made at point G on the line FA. G is located midway between F and A, although it could be any distance. Again, the example is to divide the area into two equal areas. The solution is as follows (Hashimi, 1988).

1) Compute the area of the polygon ABCDEF.

2) Compute the azimuth and length from the division point to another point on the opposite side of the polygon. In this case we compute these values for line CG.

3) Compute the area of the quadrilateral ABCG.

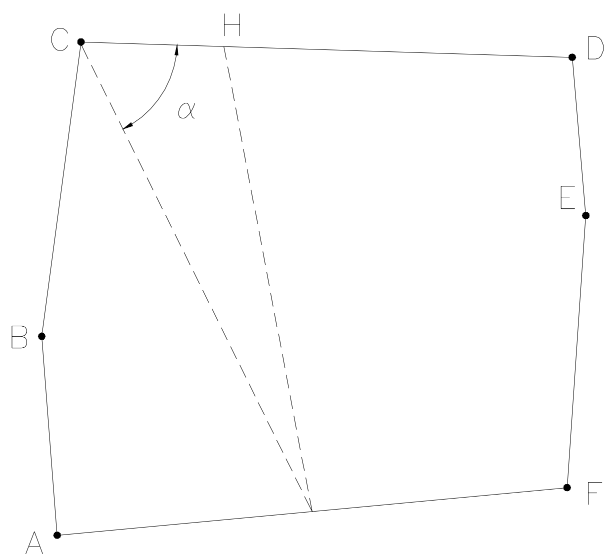

Figure 12. Example area partitioning from a point located on one sided of a polygon.

4) Since C is most probably not the desired division point, the area of the triangle CHC will define the area that needs to be added to the quadrilateral to arrive at the correct area of the new polygon.

5) Compute the distance from C to H using the relationship

$$A = \frac{1}{2} D_{GC}\ D_{CH} \sin\alpha$$

6) Check the results.

Example. Using the same values as in the previous example, we will divide the original polygon into two equal-area polygons from point G. The distance GA can be computed as half the distance between F and A. The azimuth and distance for line CG is found by adding up the latitudes and departures of lines GA, AB, and BC. Since this is a closed figure, the latitude and departure of line C must have the same magnitude with opposite sign. Thus,

$$\tan Az_{CG} = \frac{Dep_{CG}}{Lat_{CG}} \Rightarrow Az_{CG} = \tan^{-1}\left[\frac{349.0}{-818.6}\right] = 156^{o}\ 54.8'$$

$$D_{CG} = \sqrt{Dep_{CG}^2 + Lat_{CG}^2} = \sqrt{(349.0)^2 + (-818.6)^2} = 889.85'$$

The area of the initial polygon is shown as follows.

	Azimuth						
Side	Degrees	Minutes	Length	Departure	Latitude	DMD	Double Area
GA	251	53.0	316.9	-301.1	-98.5	-301.1	29670.17078
AB	340	12.0	320.4	-108.5	301.5	-710.8	-214281.3573
BC	5	38.0	618.6	60.7	615.6	-758.6	-467018.8959
CG	156	54.8	889.9	348.9	-818.6	-349.0	285650.8595
				0.0	0.0	2A =	-365979.2229
						A =	182989.6115

Since the area of the polygon is less than the desired area, we compute the area of the triangle CGH.

Required Area = 269,935 sq. ft.
A_{ABCG} = <u>182,990</u> sq. ft
A_{CGH} = 86,945 sq. ft.

Then using the formula for the area of a triangle where the area is one-half the base times the height. The height is found using the trigonometric relationship (D_{CH}) sin 60^{o} 29. Thus we have

$$86{,}945 = \frac{1}{2}\,(889.8)\,(D_{CH})\sin 60^{o}29'$$

from which the distance from C to H is found

$$D_{CH} = \frac{2\,(86{,}945)}{889.8\sin 60^{o}29'} = 224.6'$$

Check by computing the area of the polygon ABCHG.

	Azimuth						
Side	Degrees	Minutes	Length	Departure	Latitude	DMD	Double Area
GA	251	53.0	316.9	-301.1	-98.5	-301.1	29670.17078
AB	340	12.0	320.4	-108.5	301.5	-710.8	-214281.3573
BC	5	38.0	618.6	60.7	615.6	-758.6	-467018.8959
CH	96	26.0	224.6	223.2	-25.2	-474.7	11946.59976
HG	170.0	59.4	803.3	125.8	-793.4	-125.7	99751.3198
				0.0	0.0	2A =	-539932.1628
						A =	269966.0814

Again, the slight discrepancy of about 30 square feet is due to round-off errors.

Direct Division of Polygon Using Coordinates

The problem with the preceding approaches where an initial guess is made of where the dividing line occurs is that the problem changes with different polygon figures. Danial (1984) presents an approach using coordinates where the dividing line is computed directly.

Looking at figure 13, we can write the area using coordinates as follows.

$$\begin{aligned} A_{P_1P_2P_3P_4} &= A_{P'_3P_3P_4P'_4} + A_{P'_4P_4P_1P'_1} - A_{P'_1P_1P_2P'_2} - A_{P'_2P_2P_3P'_3} \\ &= \left[\frac{1}{2}(Y_3+Y_4)(X_3-X_4) + \frac{1}{2}(Y_4+Y_1)(X_4-X_1)\right] \\ &\quad - \left[\frac{1}{2}(Y_1+Y_2)(X_2-X_1) + \frac{1}{2}(Y_2+Y_3)(X_3-X_2)\right] \end{aligned}$$

The double area becomes, after manipulation

$$2A = (Y_1+Y_2)(X_1-X_2) + (Y_2+Y_3)(X_2-X_3) + (Y_3+Y_4)(X_3-X_4) + (Y_4+Y_1)(X_4-X_1)$$

If the computation is done using horizontal trapezoids off the Y-axis then the equation takes on the following form

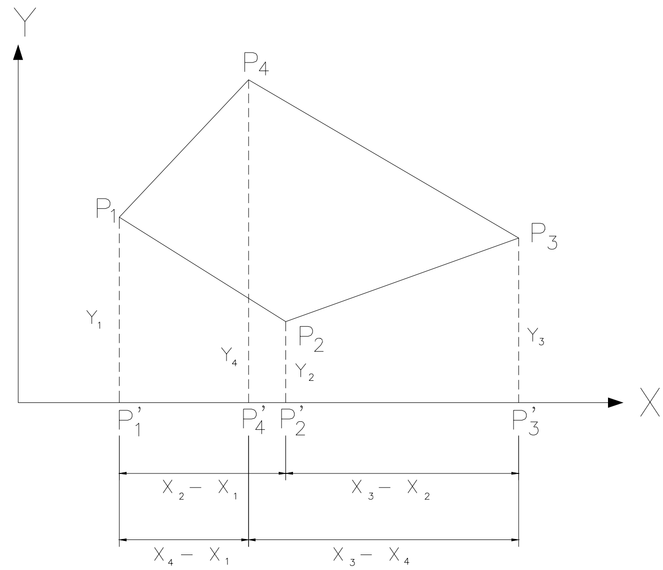

Figure 13. Geometry for computing the area of a polygon.

$$2A = (X_1 + X_2)(Y_1 - Y_2) + (X_2 + X_3)(Y_2 - Y_3) + (X_3 + X_4)(Y_3 - Y_4) + (X_4 + X_1)(Y_4 - Y_1)$$

As we know, if the area results in a negative value, the negative sign is neglected because area cannot be less than zero. The reason for the negative sign exists is because of the order in which points are selected. But, in area partitioning, it is necessary to retain the sign to ensure that the correct results are found (Danial, 1984). If the polygon area is computed clockwise then the area must be represented as a negative value.

For an n-sided polygon, the double area can be computed using the general formula:

$$\begin{aligned} 2A = \pm\big[&(Y_1 + Y_2)(X_1 - X_2) + (Y_2 + Y_3)(X_2 - X_3) + \cdots + (Y_{n-1} + Y_n)(X_{n-1} - X_n) \\ &+ (Y_n + Y_1)(X_n - X_1)\big] \end{aligned}$$

Figure 14 (from Danial, 1984) shows a situation where the area partitioning is performed using a line with a given direction. Points P_1, … P_i, P_{i+1}, … P_{n-2}, Q, and T are points on the original polygon whose coordinates are known. The dividing line, P_nP_{n-1}, has a known direction but the coordinates of the ends of the line are unknown. To solve this problem we need to solve four equations simultaneously to determine the values of the four unknowns, X_{n-1}, Y_{n-1}, X_n, and Y_n. The four equations entail three azimuth equations and one area equation.

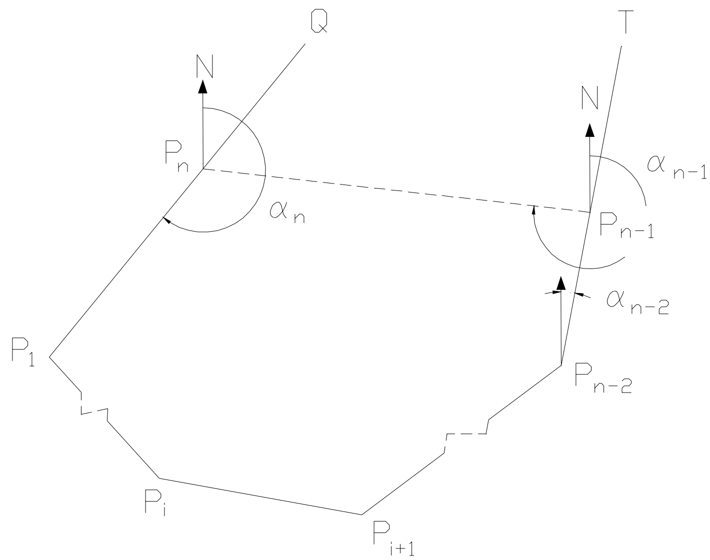

Figure 14. Area partitioning by a line with a given direction.

The three azimuth equations are (refer to figure 4):

$$\tan\alpha_{n-2} = \frac{X_{n-1} - X_{n-2}}{Y_{n-1} - Y_{n-2}}$$

$$\tan\alpha_{n-1} = \frac{X_n - X_{n-1}}{Y_n - Y_{n-1}}$$

$$\tan\alpha_n = \frac{X_1 - X_{n-1}}{Y_1 - Y_{n-1}}$$

where α_{n-2} is the azimuth of line $P_{n-2}P_{n-1}$,
α_{n-1} is the azimuth of line $P_{n-1}P_n$, and
α_n is the azimuth of line P_nP_1.

The area equation is written as:

$$2A = 2U + (Y_{n-2} + Y_{n-1})(X_{n-2} - X_{n-1}) + (Y_{n-1} + Y_n)(X_{n-1} - X_n) + (Y_n + Y_1)(X_n - X_1)$$

where 2U is the area of the remaining part of the partitioned area. It is shown as

$$2U = \sum_{i=1}^{n-3} \left(Y_i + Y_{i+1}\right)\left(X_i - X_{i+1}\right)$$

Rewrite the azimuth equations into the following form.

$$X_{n-1} - X_{n-2} = \left(Y_{n-1} - Y_{n-2}\right)\tan\alpha_{n-2}$$

$$X_n - X_{n-1} = \left(Y_n - Y_{n-1}\right)\tan\alpha_{n-1}$$

$$X_1 - X_n = \left(Y_1 - Y_n\right)\tan\alpha_n$$

Adding these three equations eliminates two unknowns, X_{n-1} and X_n. The result is

$$\begin{aligned} X_1 - X_{n-2} = &\left(\tan\alpha_{n-2} - \tan\alpha_{n-1}\right)Y_{n-1} + \left(\tan\alpha_{n-1} - \tan\alpha_n\right)Y_n \\ &+ Y_1 \tan\alpha_n - Y_{n-2}\tan\alpha_{n-2} \end{aligned}$$

Solving for one of the unknowns, like Y_n gives

$$Y_n = \frac{X_1 - X_{n-2} - Y_1 \tan\alpha_n + Y_{n-2}\tan\alpha_{n-2} - \left(\tan\alpha_{n-2} - \tan_{n-1}\right)Y_{n-1}}{\tan\alpha_{n-1} - \tan\alpha_n}$$

Designate p as a constant consisting of the known values in the equation for Y_n.

$$\boxed{p = X_1 - Y_1 \tan\alpha_n - X_{n-2} + Y_{n-2}\tan\alpha_{n-2}}$$

Then the Y-coordinate for point n becomes

$$\boxed{Y_n = \frac{p - \left(\tan\alpha_{n-2} - \tan_{n-1}\right)Y_{n-1}}{\tan\alpha_{n-1} - \tan\alpha_n}}$$

The third azimuth equation can be rewritten as

$$X_n = X_1 - Y_1 \tan\alpha_n + Y_n \tan\alpha_n$$

Substitute the value for Y_n into this equation.

$$\boxed{X_n = X_1 - Y_1 \tan\alpha_n + \frac{\tan\alpha_n}{\tan\alpha_{n-1} - \tan\alpha_n}\left[p - \left(\tan\alpha_{n-2} - \tan\alpha_{n-1}\right)Y_{n-1}\right]}$$

The first azimuth equation can be rearranged into the next formula.

$$\boxed{X_{n-1} = X_{n-2} + Y_{n-1}\tan\alpha_{n-2} - Y_{n-2}\tan\alpha_{n-2}}$$

So far we can compute three of the unknowns (Y_n, X_n, and X_{n-1}) as a function of the other unknown, Y_{n-1}.

Rewrite the double area equation by multiplying the terms on the right hand side of the equation.

$$2A = 2U + Y_{n-2}X_{n-2} + Y_{n-1}X_{n-2} - Y_{n-2}X_{n-1} + Y_nX_{n-1} - Y_{n-1}X_n + Y_1X_n - Y_nX_1 - Y_1X_1$$

Next, substitute the values for Y_n, X_n, and X_{n-1}. Lets look at each substitution individually (namely the 4th, 5th, 6th, 7th, and 8th terms on the right hand side of the double area equation).

$$Y_{n-2}X_{n-1} = Y_{n-2}X_{n-2} - Y_{n-2}^2\tan\alpha_{n-2} - Y_{n-2}\tan\alpha_{n-1}Y_{n-1}$$

$$Y_nX_{n-1} = \frac{p\left(X_{n-2} - Y_{n-2}\tan\alpha_{n-2}\right) + \left[p\tan\alpha_{n-2} - \left(\tan\alpha_{n-2} - \tan\alpha_{n-1}\right)\left(X_{n-2} - Y_{n-2}\tan\alpha_{n-2}\right)\right]Y_{n-1}}{\left(\tan\alpha_{n-1} - \tan\alpha_n\right)}$$
$$- \frac{\left(\tan\alpha_{n-2} - \tan\alpha_{n-1}\right)\tan\alpha_{n-2}Y_{n-1}^2}{\left(\tan\alpha_{n-1} - \tan\alpha_n\right)}$$

$$Y_{n-1}X_n = \left[\frac{X_1 - Y_1\tan\alpha_n + p\tan\alpha_n}{\left(\tan\alpha_{n-1} - \tan\alpha_n\right)}\right]Y_{n-1} - \frac{\left(\tan\alpha_{n-2} - \tan\alpha_{n-1}\right)\tan\alpha_n Y_{n-1}^2}{\left(\tan\alpha_{n-1} - \tan\alpha_n\right)}$$

$$Y_1X_n = Y_1X_1 - Y_1^2\tan\alpha_n + \frac{pY_1\tan\alpha_n}{\left(\tan\alpha_{n-1} - \tan\alpha_n\right)} - \frac{\left(\tan\alpha_{n-2} - \tan\alpha_{n-1}\right)Y_1\tan\alpha_n Y_{n-1}}{\left(\tan\alpha_{n-1} - \tan\alpha_n\right)}$$

If we substitute these values into the double area, one arrives at the next equation.

$$a\,Y_{n-1}^2 + b\,Y_{n-1} + c = 0$$

$$Y_nX_1 = \frac{pX_1}{\left(\tan\alpha_{n-1} - \tan\alpha_n\right)} - \frac{\left(\tan\alpha_{n-2} - \tan\alpha_{n-1}\right)X_1Y_{n-1}}{\left(\tan\alpha_{n-1} - \tan\alpha_n\right)}$$

where the coefficients are defined as (Danial, 1984)[2]:

$$a = \frac{\left(\tan\alpha_{n-2} - \tan\alpha_{n-1}\right)\left(\tan\alpha_n - \tan\alpha_{n-2}\right)}{\tan\alpha_{n-1} - \tan\alpha_n}$$

$$b = 2p\,\frac{\tan\alpha_{n-2} - \tan\alpha_{n-1}}{\tan\alpha_{n-1} - \tan\alpha_n}$$

$$c = \frac{p\left(X_{n-2} - Y_{n-2}\tan\alpha_{n-2} + Y_1\tan\alpha_n - X_1\right)}{\tan\alpha_{n-1} - \tan\alpha_n} - Y_1^2\tan\alpha_n + Y_{n-2}^2\tan\alpha_{n-2} + 2U - 2A$$

The solution for Y_{n-1} is found using the quadratic equation.

$$Y_{n-1} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Using the example from the previous section dealing with the estimated partitioning using line HI, which is parallel to line AF, the results using the method presented by Danial is given in the following Mathcad program. Designate H as P_n and I as P_{n-1}.

Solution to Area Partitioning Problem Using the Method Presented by Danial

```
dd(ang) := | degree ← floor(ang)
           | mins ← (ang − degree)·100.0
           | minutes ← floor(mins)
           | seconds ← (mins − minutes)·100.0
           | degree + minutes/60.0 + seconds/3600.0

radians(ang) := | d ← dd(ang)
                | d·π/180.0

dms(ang) := | degree ← floor(ang)
            | rem ← (ang − degree)·60
            | mins ← floor(rem)
            | rem1 ← (rem − mins)
            | secs ← rem1·60.0
            | degree + mins/100 + secs/10000

todeg := 180/π
```

---

[2] Note that Danial gives the formula for the coefficient c as follows:

$$c = -\frac{p}{\tan\alpha_{n-1} - \tan\alpha_n} - Y_1^2\tan\alpha_n + Y_{n-2}^2\tan\alpha_{n-2} + 2U - 2A$$

Instead, the formula should be presented as given in the text of this paper and shown above.

$$c = \frac{p\left(X_{n-2} - Y_{n-2}\tan\alpha_{n-2} + Y_1\tan\alpha_n - X_1\right)}{\tan\alpha_{n-1} - \tan\alpha_n} - Y_1^2\tan\alpha_n + Y_{n-2}^2\tan\alpha_{n-2} + 2U - 2A$$

The given values for this problem are:

$X_A := 1000.0$ $\quad Y_A := 500.0$ $\quad Az_{FE} := 2.27$

$X_B := 891.5$ $\quad Y_B := 801.5$ $\quad \alpha_{n2} := \text{radians}(Az_{FE})$

$X_C := 952.2$ $\quad Y_C := 1417.1$ $\quad Az_{IH} := 251.53$

$X_D := 1602.4$ $\quad Y_D := 1343.8$ $\quad \alpha_{n1} := \text{radians}(Az_{IH})$

$X_E := 1621.4$ $\quad Y_E := 1144.1$ $\quad Az_{CB} := 185.38$

$X_F := 1602.3$ $\quad Y_F := 697.0$ $\quad \alpha_n := \text{radians}(Az_{CB})$

$A := 269935$

$X_{n2} := X_F$ $\quad Y_{n2} := Y_F$

$X_1 := X_B$ $\quad Y_1 := Y_B$

$X_1 := X_B$ $\quad Y_1 := Y_B$

The solution is as follows (Note that the subscript n1 is shorthand for n-1 and the subscript n2 is shorthand for n-2:

$$U := \frac{1}{2}\cdot\left[(Y_B + Y_A)\cdot(X_B - X_A) + (Y_A + Y_F)\cdot(X_A - X_F)\right] \qquad U = -431082.925$$

$$p := X_1 - Y_1\cdot\tan(\alpha_n) - X_{n2} + Y_{n2}\cdot\tan(\alpha_{n2}) \qquad p = -760.1932$$

$$a := \frac{(\tan(\alpha_{n2}) - \tan(\alpha_{n1}))\cdot(\tan(\alpha_n) - \tan(\alpha_{n2}))}{\tan(\alpha_{n1}) - \tan(\alpha_n)} \qquad a = -0.05711$$

$$b := 2p\cdot\frac{\tan(\alpha_{n2}) - \tan(\alpha_{n1})}{\tan(\alpha_{n1}) - \tan(\alpha_n)} \qquad b = 1549.1977$$

$$c := \frac{p\cdot(X_{n2} - Y_{n2}\cdot\tan(\alpha_{n2}) + Y_1\cdot\tan(\alpha_n) - X_1)}{\tan(\alpha_{n1}) - \tan(\alpha_n)} - Y_1^{\,2}\cdot\tan(\alpha_n) + Y_{n2}^{\,2}\cdot\tan(\alpha_{n2}) + 2U - 2A$$

$$c = -1640129.8624$$

The Solve function is used to arrive at the solution. Two possible answers are possible. The logical answer is assigned to the value for Y_{n-1}, shown here as Y_{n1}.

$$a\cdot x^2 + b\cdot x + c \text{ solve}, x \rightarrow \begin{pmatrix} 1103.5930070219402537 \\ 26112.676551174948972 \end{pmatrix}$$

Next, the remaining unknown coordinates are computed as follows:

$$X_{n1} := X_{n2} + Y_{n1}\cdot\tan(\alpha_{n2}) - Y_{n2}\cdot\tan(\alpha_{n2}) \qquad X_{n1} = 1619.6966$$

$$X_n := X_1 - Y_1\cdot\tan(\alpha_n) + \frac{\tan(\alpha_n)}{\tan(\alpha_{n1}) - \tan(\alpha_n)}\cdot\left[p - (\tan(\alpha_{n2}) - \tan(\alpha_{n1}))\cdot Y_{n1}\right]$$

$$Y_n := \frac{p - (\tan(\alpha_{n2}) - \tan(\alpha_{n1}))\cdot Y_{n1}}{\tan(\alpha_{n1}) - \tan(\alpha_n)}$$

$$X_n = 898.0208$$

$$Y_n = 867.4776$$

Check the results. As a first check, the azimuth of the dividing line should be equal to the line AF. Hence,

$$Az_{IH} = \tan^{-1}\left(\frac{X_H - X_I}{Y_H - Y_I}\right) = \tan^{-1}\left(\frac{898.01 - 1619.7}{867.47 - 1103.59}\right)$$
$$= 251^\circ\ 52'\ 59''$$

The azimuth checks with the azimuth from F to A. Next, check the area to ensure that it is correct. The results are presented in the following spreadsheet. Note that the third column is the difference X_i - X_{i+1}, the fourth column is the sum Y_i + Y_{i+1}, and the fifth column is the product of the difference and the sum. From the results, it is obvious that this method can be used to easily partition an area using a line with a given direction. The difference between the exact value and the computed value is only about 3.4 square feet. This difference is attributed to round-off problems.

STA	X	Y			
A	1000.00	500.00	108.50	1301.50	141212.75
B	891.50	801.50	-6.51	1668.97	-10864.99
H	898.01	867.47	-721.69	1971.06	-1422494.29
I	1619.70	1103.59	17.40	1800.59	31330.27
F	1602.30	697.00	602.30	1197.00	720953.10
A	1000.00	500.00			
					-539863.17
				Area =	269931.59

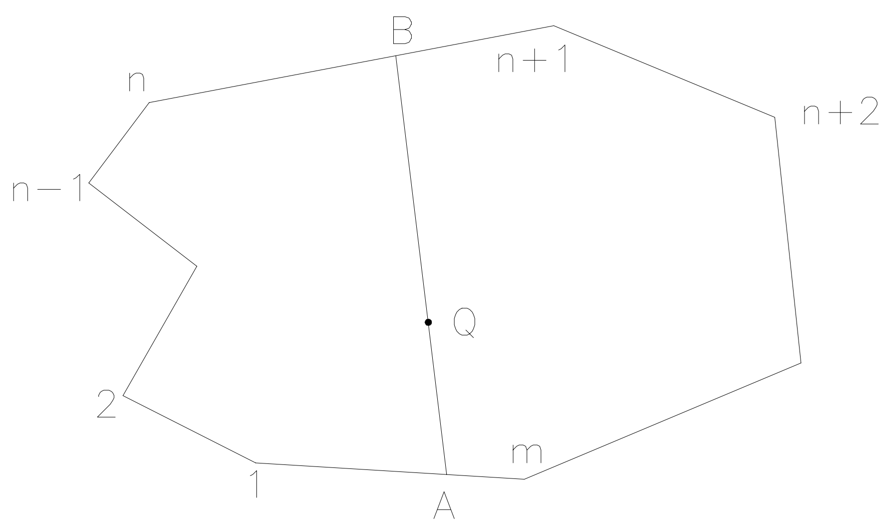

Figure 15. Example polygon showing the dividing line used in the development by Easa (1989).

Easa (1989) presents a very similar direct approach to area partitioning using coordinates. Using the azimuth relationship (refer to figure 15), the Y-coordinates for the ends of the dividing line can be written in the following form

$$Y_A = Y_1 + (X_A - X_1)\cot\alpha_{1m}$$

$$Y_B = Y_n + (X_B - X_n)\cot\alpha_{n,n+1}$$

Following the notation from Easa, designate S as the cotangent of the azimuth. We can then rewrite these equations as

$$\boxed{Y_A = Y_1 + (X_A - X_1)S_A}$$

$$\boxed{Y_B = Y_n + (X_B - X_n)S_B}$$

where,

$$S_A = \cot\alpha_{1m}$$
$$S_B = \cot\alpha_{n,n+1}$$

Also, recall that the area of polygon A, 1, 2, …, n, B can be written as

$$A = \frac{1}{2}\left[X_B(Y_n - Y_A) + X_A(Y_B - Y_1) + X_1 Y_A - X_n Y_B + \sum_{i=1}^{n-1}(X_{i+1}Y_i - X_i Y_{i+1})\right]$$

These last three equations form the basic design formulas for area partitioning as identified by Easa (1989). There are four unknowns, which means that a fourth equation is required to solve the problem. This last equation is based on the nature of the dividing line. In essence, Easa also uses three azimuth equations and an area equation for the solution to the area-partitioning problem.

Lets look at the same situation already presented by Danial where the dividing line is fixed in direction. Following after Easa, designate the cotangent of the azimuth of the dividing line with P (direction from A to B). If the dividing line is parallel to another line in the polygon defined by the end points u and v, then the cotangent of the azimuth of that dividing line can be defined as

$$P = \frac{Y_v - Y_u}{X_v - X_u}$$

If the dividing line is perpendicular to the line uv, then the relationship is written as

$$P = -\frac{X_v - X_u}{Y_v - Y_u}$$

Write the equation for the cotangent of the azimuth of the dividing line in terms of the four unknowns.

$$P = \frac{Y_B - Y_A}{X_B - X_A}$$

Substitute the values for Y_A and Y_B.

$$P = \frac{Y_n + (X_B - X_n)S_B - Y_1 - (X_A - X_1)S_A}{X_B - X_A}$$

Rearrange the equation to in a form for solving X_A.

$$X_B P - X_A P = Y_n + X_B S_B - X_n S_B - Y_1 - X_A S_A + X_1 S_A$$

$$-X_A P + X_A S_A = -X_B P + X_B S_B + Y_n - X_n S_B - Y_1 + X_1 S_A$$

$$X_A = \left(\frac{P - S_B}{P - S_A}\right) + \frac{-(Y_n - Y_1 + X_1 S_A - X_n S_B)}{P - S_A}$$

Designating

$$k_2 = \frac{-(Y_n - Y_1 + X_1 S_A - X_n S_B)}{P - S_A} \quad \text{and}$$

$$k_3 = \frac{P - S_B}{P - S_A}$$

the equation for X_A can be rewritten as

$$\boxed{X_A = k_2 + k_3 X_B}$$

If the dividing line is along the meridian, or nearly so, then $P = \infty$ and $k_2 = 0$ and $k_3 = 1$. Then the unknowns can be easily solved by simultaneous equations using the area formula, the two azimuth relationships (Y_A and Y_B), and the formula above for X_A.

Substitute the two equations for Y_A and Y_B into the area equation.

$$A = \frac{1}{2}\{X_B[Y_n - Y_1 - (X_A - X_1)S_A] + X_A[Y_n + (X_B - X_n)S_B - Y_1] + X_1Y_1$$

$$+ X_1(X_A - X_1)S_A - X_nY_n - X_n(X_B - X_n)S_B + \sum_{i=1}^{n-1}(X_{i+1}Y_i - X_iY_{i+1})\}$$

Designate $k_1 = \left[\sum_{i=1}^{n-1}(X_{i+1}Y_i - X_iY_{i+1})\right] - 2A$. Rearrange the area formula.

$$X_BY_n - X_BY_1 - X_B(X_A - X_1)S_A + X_AY_n + X_A(X_B - X_n)S_B - X_AY_1 + X_AX_1S_A$$
$$- X_BX_nS_B + X_1Y_1 - X_1^2S_A - X_nY_n + X_n^2S_B + k_1 = 0$$

$$(S_B - S_A)X_AX_B + (X_A + X_B)(Y_n - Y_1 + X_1S_A - X_nS_B)$$
$$+ (X_1Y_1 - X_1^2S_A - X_nY_n + X_n^2S_B + k_1) = 0$$

Let

$$k_4 = X_1Y_1 - X_1^2S_A - X_nY_n + X_n^2S_B + k_1$$
$$k_5 = Y_N - Y_1 + X_1S_A - X_nS_B$$
$$k_6 = S_B - S_A$$

Then we have the following formula

$$k_4 + k_5(X_A + X_B) + k_6X_AX_B = 0$$

Substitute the value for X_A into the equation.

$$k_4 + k_5(k_2 + k_3X_B + X_B) + k_6(k_2 + k_3X_B)X_B = 0$$

$$k_3k_6X_B^2 + (k_3k_5 + k_5 + k_2k_6)X_B + (k_4 + k_2k_5) = 0$$

This is the quadratic form which can be succinctly written as

$$\boxed{aX_B^2 + bX_B + c = 0}$$

where,

$$a = k_3k_6$$
$$b = k_3k_5 + k_5 + k_2k_6$$
$$c = k_4 + k_2k_5$$

Solve for X_B using the quadratic equation. Then back substitute into the equations for X_A, Y_A, and Y_B. The same example used before for dividing a polygon with a line parallel to a line on the edge of the polygon is given in the following Mathcad program.

Solution to Area Partitioning Problem Using the Method Presented by Easa

$$\mathrm{dd}(ang) := \begin{array}{|l} degree \leftarrow \mathrm{floor}(ang) \\ mins \leftarrow (ang - degree)\cdot 100.0 \\ minutes \leftarrow \mathrm{floor}(mins) \\ seconds \leftarrow (mins - minutes)\cdot 100.0 \\ degree + \dfrac{minutes}{60.0} + \dfrac{seconds}{3600.0} \end{array}$$

$$\mathrm{radians}(ang) := \begin{array}{|l} d \leftarrow \mathrm{dd}(ang) \\ d\cdot\dfrac{\pi}{180.0} \end{array}$$

$$todeg := \frac{180}{\pi}$$

$$\mathrm{dms}(ang) := \begin{array}{|l} degree \leftarrow \mathrm{floor}(ang) \\ rem \leftarrow (ang - degree)\cdot 60 \\ mins \leftarrow \mathrm{floor}(rem) \\ rem1 \leftarrow (rem - mins) \\ secs \leftarrow rem1\cdot 60.0 \\ degree + \dfrac{mins}{100} + \dfrac{secs}{10000} \end{array}$$

The given values for this problem are:

$X_A := 1000.0$ $\quad Y_A := 500.0$

$X_B := 891.5$ $\quad Y_B := 801.5$

$X_C := 952.2$ $\quad Y_C := 1417.1$

$X_D := 1602.4$ $\quad Y_D := 1343.8$

$X_E := 1621.4$ $\quad Y_E := 1144.1$

$X_F := 1602.3$ $\quad Y_F := 697.0$

$A := 269935$

$Az_{FA} := 251.53$

$$P := \frac{1}{\tan(\mathrm{radians}(Az_{FA}))}$$

$Az_{FE} := 2.27$

$$S_A := \frac{1}{\tan(\mathrm{radians}(Az_{FE}))}$$

$Az_{BC} := 5.38$

$$S_B := \frac{1}{\tan(\mathrm{radians}(Az_{BC}))}$$

$X_n := X_B$ $\quad Y_n := Y_B$

$X_1 := X_F$ $\quad Y_1 := Y_F$

Solution: First compute the constants k's

$k_1 := (X_A \cdot Y_F - X_F \cdot Y_A) + (X_B \cdot Y_A - X_A \cdot Y_B) - 2A$ $\quad k_1 = -999770.00$

$k_2 := \dfrac{-(Y_n - Y_1 + X_1 \cdot S_A - X_n \cdot S_B)}{P - S_A}$ $\quad k_2 = 1238.16$

$k_3 := \dfrac{P - S_B}{P - S_A}$ $\quad k_3 = 0.42$

$k_4 := X_1 \cdot Y_1 - X_1^2 \cdot S_A - X_n \cdot Y_n + X_n^2 \cdot S_B + k_1$ $\quad k_4 = -52559913.68$

$k_5 := Y_n - Y_1 + X_1 \cdot S_A - X_n \cdot S_B$ $\quad k_5 = 28532.93$

$k_6 := S_B - S_A$ $\quad k_6 = -13.25$

$a := k_3 \cdot k_6$

$b := k_3 \cdot k_5 + k_5 + k_2 \cdot k_6$

$c := k_4 + k_2 \cdot k_5$

$$a \cdot x^2 + b \cdot x + c \text{ solve}, x \rightarrow \begin{pmatrix} 898.00778884360475364 \\ 3411.4405649423651734 \end{pmatrix}$$

Designate the X and Y coordinate values in terms of Eastings and Northings to avoid the confusion of using the values for already assigned to the traverse. Thus, $E_A = X_A$ in the text of this paper. $E_A = X_I$. Also, $E_B = X_H$. The same can be said for the Northings.

$E_B := 898.008$ $\qquad X_H := E_B$

$E_A := k_2 + k_3 \cdot E_B$ $\qquad X_I := E_A$

$N_A := Y_1 + (E_A - X_1) \cdot S_A$ $\qquad Y_I := N_A$

$N_B := Y_n + (E_B - X_n) \cdot S_B$ $\qquad Y_H := N_B$

The results are:

$X_H = 898.01$ $\qquad Y_H = 867.35$

$X_I = 1619.69$ $\qquad Y_I = 1103.46$

The other common problem that has already been identified is the division of an area from a defined point on the boundary of the polygon. Again, Danial (1984) presents a direct method to solve this problem using coordinates. Since one of the end points of the dividing line is known, only two equations need to be solved simultaneously. This involves an azimuth equation and the area equation. The azimuth equation can be written as (refer to figure 16)

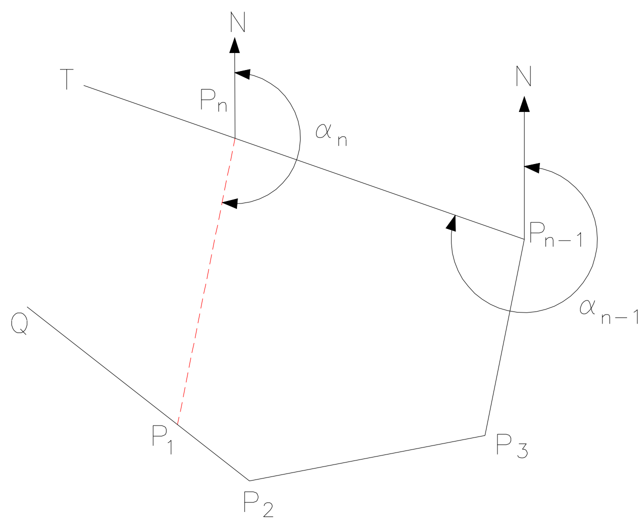

Figure 16. Example for area partitioning after the developments by Danial (1984).

$$\boxed{X_n = X_{n-1} + (Y_n - Y_{n-1}) \tan \alpha_{n-1}}$$

The area equation was presented earlier as

$$2A = 2U + (Y_{n-1} + Y_n)(X_{n-1} - X_n) + (Y_n + Y_1)(X_n - X_1)$$

where $2U = \sum_{i=1}^{n-2} (Y_i + Y_{i+1})(X_i - X_{i+1})$.

Rearrange the double area formula.

$$2A = 2U + X_n (Y_1 - Y_{n-1}) - Y_n (X_1 - X_{n-1}) + Y_{n-1} X_{n-1} - Y_1 X_1$$

Substitute the value for X_n from the azimuth equation.

$$2A = 2U + [X_{n-1} + (Y_n - Y_{n-1}) \tan \alpha_{n-1}](Y_1 - Y_{n-1}) - Y_n (X_1 - X_{n-1}) + Y_{n-1} X_{n-1} - Y_1 X_1$$

$$= 2U + Y_1 X_{n-1} - Y_{n-1} X_{n-1} + Y_1 (Y_n - Y_{n-1}) \tan \alpha_{n-1} - Y_{n-1} (Y_n - Y_{n-1}) \tan \alpha_{n-1}$$
$$Y_n (X_1 - X_{n-1}) + Y_{n-1} X_{n-1} - Y_1 X_1$$

$$= 2U - Y_n (Y_{n-1} \tan \alpha_{n-1} - Y_1 \tan \alpha_{n-1} + X_1 - X_{n-1}) - Y_1 (X_1 - X_{n-1})$$
$$- Y_{n-1} (Y_1 - Y_{n-1}) \tan \alpha_{n-1}$$

Solve for Y_n.

$$Y_n\left[(X_1 - X_{n-1}) - (Y_1 - Y_{n-1})\tan\alpha_{n-1}\right] = 2U - 2A - Y_1(X_1 - X_{n-1}) - Y_{n-1}(Y_1 - Y_{n-1})\tan\alpha_{n-1}$$

$$\boxed{Y_n = \frac{2U - 2A - Y_1(X_1 - X_{n-1}) - Y_{n-1}(Y_1 - Y_{n-1})\tan\alpha_{n-1}}{(X_1 - X_{n-1}) - (Y_1 - Y_{n-1})\tan\alpha_{n-1}}}$$

Following is the same example that was used by the estimated division method. As one can see with the results of the area computations given at the end of the Mathcad program, the correct area is accurately determined.

Solution to Area Partitioning Problem Using the Method Presented by Danial

```
dd(ang) := | degree ← floor(ang)
           | mins ← (ang − degree)·100.0
           | minutes ← floor(mins)
           | seconds ← (mins − minutes)·100.0
           | degree + minutes/60.0 + seconds/3600.0

radians(ang) := | d ← dd(ang)
                | d·π/180.0

dms(ang) := | degree ← floor(ang)
            | rem ← (ang − degree)·60
            | mins ← floor(rem)
            | rem1 ← (rem − mins)
            | secs ← rem1·60.0
            | degree + mins/100 + secs/10000

todeg := 180/π
```

The given values for this problem are:

$X_A := 1000.0$ $Y_A := 500.0$

$X_B := 891.5$ $Y_B := 801.5$

$X_C := 952.2$ $Y_C := 1417.1$ $Az_{CD} := 96.26$

$X_D := 1602.4$ $Y_D := 1343.8$ $\alpha_{n1} := \text{radians}(Az_{CD})$

$X_E := 1621.4$ $Y_E := 1144.1$

$X_F := 1602.3$ $Y_F := 697.0$

$A := -269935$ The area is negative because the determination of the area is clockwise in this problem.

The solution is as follows (Note that the subscript n1 is shorthand for n-1)

The point to be held fixed, G, is the point midway between points F and A.

$$X_G := \frac{X_F + X_A}{2} \qquad Y_G := \frac{Y_F + Y_A}{2}$$

$X_G = 1301.15$

$Y_G = 598.50$

$$U := \frac{1}{2}\cdot\left[\left(Y_G + Y_A\right)\cdot\left(X_G - X_A\right) + \left(Y_A + Y_B\right)\cdot\left(X_A - X_B\right) + \left(Y_B + Y_C\right)\cdot\left(X_B - X_C\right)\right]$$

$U = 168678.503$

$X_1 := X_G \qquad Y_1 := Y_G \qquad X_{n1} := X_C \qquad Y_{n1} := Y_C$

$$Y_n := \frac{2U - 2A - Y_1\cdot\left(X_1 - X_{n1}\right) - Y_{n1}\cdot\left(Y_1 - Y_{n1}\right)\cdot\tan\left(\alpha_{n1}\right)}{\left(X_1 - X_{n1}\right) - \left(Y_1 - Y_{n1}\right)\cdot\tan\left(\alpha_{n1}\right)}$$

$$X_n := X_{n1} + \left(Y_n - Y_{n1}\right)\cdot\tan\left(\alpha_{n1}\right)$$

$X_H := X_n \qquad Y_H := Y_n$

$X_H = 1175.34 \qquad Y_H = 1391.94$

Check the results to ensure that the area is correct. The results are given in the following table.

STA	X	Y			
A	1000.00	500.00	108.50	1301.50	141212.75
B	891.50	801.50	-60.70	2218.60	-134669.02
C	952.20	1417.10	-223.14	2809.04	-626809.19
H	1175.34	1391.94	-125.81	1990.44	-250417.26
G	1301.15	598.50	301.15	1098.50	330813.28
A	1000.00	500.00			
					-539869.44
				Area =	269934.72

As we have seen before, Easa (1989) uses the same method of partitioning a polygon using coordinates. Again, the approach is a little different. Recall that the azimuth equation was written in the next form.

$$Y_B = Y_n + \left(X_B - X_n\right)S_B$$

where the subscript B indicates the unknown end of the dividing line (point A is fixed on this line). S_B is the azimuth of the line on the polygon where point B will be located (see figure 15). Also recall that the area equation was given as

$$A = \frac{1}{2}\left[X_B\left(Y_n - Y_A\right) + X_A\left(Y_B - Y_1\right) + X_1 Y_A - X_n Y_B + \sum_{i=1}^{n-1}\left(X_{i+1}Y_i - X_i Y_{i+1}\right)\right]$$

Further, the Easa defined the constant k_1 as

$$k_1 = \left[\sum_{i=1}^{n-1}\left(X_{i+1}Y_i - X_i Y_{i+1}\right)\right] - 2A$$

Thus, the area equation can be written as

$$X_B\left(Y_n - Y_A\right) + X_A\left(Y_B - Y_1\right) + X_1 Y_A - X_n Y_B + k_1 = 0$$

Substitute the value for Y_B from the azimuth equation into the area equation and expand the equation.

$$X_B Y_n - X_B Y_A + X_A\left[Y_n + \left(X_B - X_n\right)S_B\right] - X_A Y_1 + X_1 Y_A - X_n\left[Y_n + \left(X_B - X_n\right)S_B\right] + k_1 = 0$$

$$\left[\left(Y_A - Y_n\right) - \left(X_A - X_n\right)S_B\right]X_B = \left(Y_n - X_n S_B - Y_1\right)X_A - \left(Y_n - X_n S_B\right)X_n + X_1 Y_A + k_1$$

Then, solve for X_B.

$$X_B = \frac{k_1 - X_A Y_1 + X_1 Y_A + \left(X_A - X_n\right)\left(Y_n - X_n S_B\right)}{\left(Y_A - Y_n\right) - \left(X_A - X_n\right)S_B}$$

Next, solve for the Y-coordinate using the azimuth equation. If the fixed point is B and A is the unknown on the dividing line, the solution uses the following form:

$$X_A = \frac{k_1 - X_B Y_n + X_n Y_B + \left(X_B - X_1\right)\left(Y_1 - X_1 S_A\right)}{\left(Y_B - Y_1\right) - \left(X_B - X_1\right)S_A}$$

and

$$Y_A = Y_1 + \left(X_A - X_1\right)S_A$$

where S_A is the cotangent of the azimuth of the line on the original polygon where A will be placed. Using the same example that was presented previously, the results are depicted in the following Mathcad program. The results for X_H and Y_H are the same as found using the approach by Danial.

Solution to Area Partitioning Problem Using the Method Presented by Easa

$$dd(ang) := \begin{vmatrix} degree \leftarrow floor(ang) \\ mins \leftarrow (ang - degree)\cdot 100.0 \\ minutes \leftarrow floor(mins) \\ seconds \leftarrow (mins - minutes)\cdot 100.0 \\ degree + \dfrac{minutes}{60.0} + \dfrac{seconds}{3600.0} \end{vmatrix}$$

$$radians(ang) := \begin{vmatrix} d \leftarrow dd(ang) \\ d\cdot\dfrac{\pi}{180.0} \end{vmatrix}$$

$$todeg := \frac{180}{\pi}$$

$$dms(ang) := \begin{vmatrix} degree \leftarrow floor(ang) \\ rem \leftarrow (ang - degree)\cdot 60 \\ mins \leftarrow floor(rem) \\ rem1 \leftarrow (rem - mins) \\ secs \leftarrow rem1\cdot 60.0 \\ degree + \dfrac{mins}{100} + \dfrac{secs}{10000} \end{vmatrix}$$

The given values for this problem are:

$X_A := 1000.0$ $\quad Y_A := 500.0$

$X_B := 891.5$ $\quad Y_B := 801.5$

$X_C := 952.2$ $\quad Y_C := 1417.1$ $\quad Az_{CD} := 96.26$

$X_D := 1602.4$ $\quad Y_D := 1343.8$ $\quad \alpha_{n1} := radians(Az_{CD})$

$X_E := 1621.4$ $\quad Y_E := 1144.1$ $\quad S_B := \dfrac{1}{\tan(\alpha_{n1})}$

$X_F := 1602.3$ $\quad Y_F := 697.0$

$A := 269935$

The solution is as follows:

The point to be held fixed, G, is the point midway between points F and A.

$$X_G := \frac{X_F + X_A}{2} \qquad Y_G := \frac{Y_F + Y_A}{2}$$

$X_G = 1301.15$

$Y_G = 598.50$

Note that the lower case variables, x and y, with subscripts A and B represent the ends of the dividing line presented in the paper.

$x_A := X_G$ $\quad y_A := Y_G$ $\quad X_n := X_C$ $\quad Y_n := Y_C$ $\quad X_1 := X_A$ $\quad Y_1 := Y_A$

$$k_1 := \left(X_B \cdot Y_A - X_A \cdot Y_B\right) + \left(X_C \cdot Y_B - X_B \cdot Y_C\right) - 2A$$

$$k_1 = -1395776.35$$

$$x_B := \frac{k_1 - x_A \cdot Y_1 + X_1 \cdot y_A + \left(x_A - X_n\right) \cdot \left(Y_n - X_n \cdot S_B\right)}{\left(y_A - Y_n\right) - \left(x_A - X_n\right) \cdot S_B}$$

$$y_B := Y_n + \left(x_B - X_n\right) \cdot S_B$$

$$X_H := x_B \qquad Y_H := y_B$$

$$X_H = 1175.34$$

$$Y_H = 1391.94$$

Easa (1989) also presents a solution to the problem of placing the dividing line through an interior point of the polygon. Since there are an infinite number of dividing lines that can pass through this point (for example, point Q in figure 15), the edge of the polygon where the dividing line will fall (A and B in figure 5) needs to be identified.

We can write the relationship between Q and the end points A and B as

$$\frac{Y_A - Y_B}{X_A - X_B} = \frac{Y_Q - Y_B}{X_Q - X_B}$$

Recall that earlier the Y-coordinate of the ends of the dividing line were presented as

$$Y_A = Y_1 + \left(X_A - X_1\right) S_A$$

$$Y_B = Y_n + \left(X_B - X_n\right) S_B$$

Substitute these two formulas into the previous equation.

$$\frac{Y_1 + X_A S_A - X_1 S_A - Y_n - X_B S_B + X_n S_B}{X_A - X_B} = \frac{Y_Q - Y_n - X_B S_B + X_n S_B}{X_Q - X_B}$$

$$\left(X_Q S_A - Y_Q + Y_n - X_n S_B\right) X_A + \left(Y_Q - Y_1 + X_1 S_A - X_Q S_B\right) X_B + \left(S_B - X_A\right) X_A X_B + \left(Y_1 - X_1 S_A - Y_n + X_n S_B\right) X_Q = 0$$

Following Easa, let

$$k_7 = X_Q S_A - Y_Q + Y_n - X_n S_B,$$
$$k_8 = Y_Q - Y_1 + X_1 S_A - X_Q S_B, \quad \text{and}$$
$$k_9 = \left(Y_1 - Y_n + X_n S_B - X_1 S_A\right) X_Q = -k_5 X_Q$$

Remember that

$$k_6 = S_B - S_A$$

Then,

$$k_7 X_A + k_8 X_B + k_6 X_A X_B + k_9 = 0$$

But earlier we saw that

$$k_4 + k_5\left(X_A + X_B\right) + k_6 X_A X_B = 0$$

Substitute the value for the product $X_A X_B$ into the previous equation. This results in

$$k_7 X_A + k_8 X_B - k_4 - k_5 X_A - k_5 X_B + k_9 = 0$$

Solving for X_A yields

$$X_A = \frac{\left(k_4 - k_9\right) + \left(k_5 - k_8\right) X_B}{k_7 - k_5}$$

Substitute this back into the equation before ($k_4 + k_5\left(X_A + X_B\right) + k_6 X_A X_B = 0$) which leads to

$$k_4 + \frac{k_5\left(k_4 - k_9\right) + k_5\left(k_5 - k_8\right) X_B}{k_7 - k_5} + k_5 X_B + \frac{k_6 X_B\left(k_4 - k_9\right) + k_6\left(k_5 - k_8\right) X_B^2}{k_7 - k_5} = 0$$

$$k_6\left(k_5 - k_8\right) X_B^2 + \left[k_5\left(k_7 - k_8\right) + k_6\left(k_4 - k_9\right)\right] X_B + \left(k_4 k_7 - k_5 k_9\right) = 0$$

Partitioning Pie-Shaped Polygons

Up until now, the area partitioning problem discussed the method of dividing a polygon consisting of a boundary of straight lines. For various reasons, there are numerous situations where one or more of the boundaries consist of a curve. One set of derivations for pie-shaped lots was developed by Danial (1990).

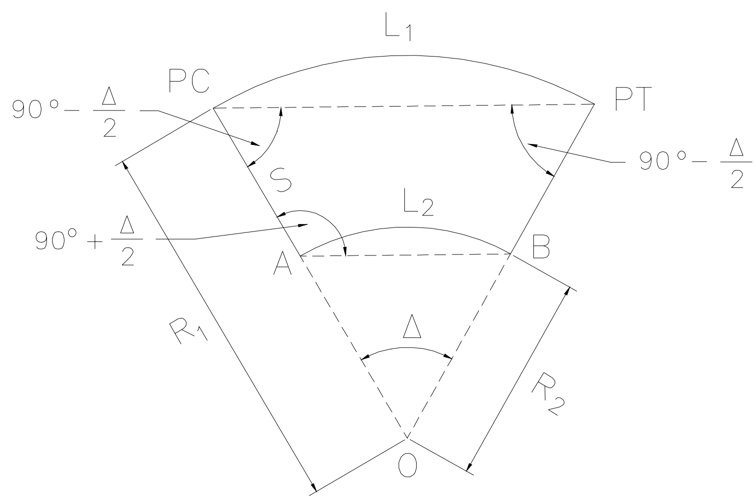

Figure 17. Area computation on a pie-shaped polygon.

Figure 17 shows the geometry of a pie-shaped parcel. The distance S from A to the PC is found as

$$S = D_{O-PC} - D_{O-A} = R_1 - R_2$$

From the principles of horizontal curve geometry, the arc lengths become

$$L_1 = R_1\Delta$$

$$L_2 = R_2\Delta$$

where Δ is in radians. We also know from curve geometry that the angle at the PC from the chord to the center of the circle is $90^o - \Delta/2$. Since AB is parallel to the long chord, then the angle $\angle_{PC-A-B} = 90^o + \Delta/2$. The area of arc 1 (PC-PT-O) is found as

$$A_{PC-PT-O} = \frac{1}{2}R_1^2\Delta$$

while the area of the second arc (A-B-O) is

$$A_{A-B-O} = \frac{1}{2}R_2^2\Delta$$

where Δ is again in radians. The area of the lot defined by A-PC-PT-B is the difference in the two areas.

$$A_{A-PC-PT-B} = \frac{1}{2}\left(R_1^2 - R_2^2\right)\Delta$$

The problem of dividing the pie-shaped lot can be broken down into two different forms, just as we have seen before. These are partitioning by a random line through a point and partitioning by a line parallel to one of the radial sides. Lets look at the former situation first.

Case 1 (Danial, 1990) involves dividing parcel PC-PT-A-B into two polygons. The arc length from the PC to C (figure 18) is L_{11} while the arc length from A to D is L_{21}. These lengths are known. Solving for the area of polygon PC-C-D-A and C-PT-B-D commences as follows.

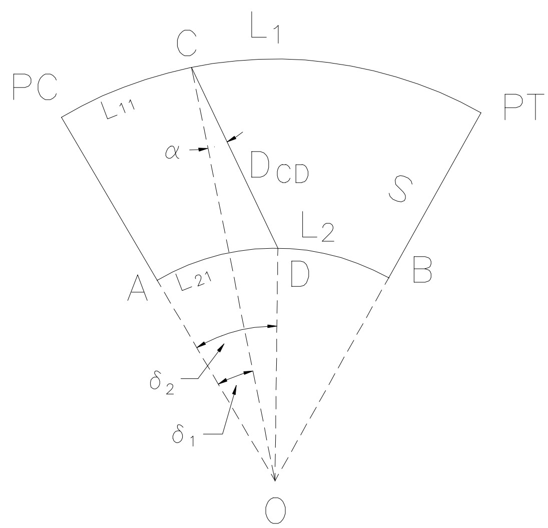

Figure 18. Area partitioning of a pie-shaped figure through a point.

The central angle between the PC and C (δ_1) and between A and D (δ_2) are found from

$$\delta_1 = {L_{11}}/{R_1}$$

$$\delta_2 = {L_{21}}/{R_2}$$

from which $\angle_{COD} = \delta_2 - \delta_1$. Using the law of cosines,

$$D_{CD} = \left[R_1^2 + R_2^2 - 2R_1R_2 \cos\angle_{COD}\right]^{1/2}$$

Then, from the law of sines,

$$\alpha = \sin^{-1}\left(\frac{\sin \angle_{COD}}{D_{CD}} R_2 \right)$$

The angle at D then becomes

$$\angle_{ODC} = 180^{o} - \left(\alpha + \angle_{COD} \right)$$

The area of parcel PC-C-D-A is found by computing the area of the sector PC-O-C and adding the area of triangle C-O-D and subtracting the area of sector O-A-D.

$$A_{PC-C-D-A} = A_{PC-O-C} + A_{O-C-D} - A_{O-A-D}$$

Substituting the formulas for the different areas, this becomes

$$A_{PC-C-D-A} = \frac{1}{2} R_1^2 \delta_1 + \frac{1}{2} R_1 R_2 \sin \angle_{COB} - \frac{1}{2} R_2^2 \delta_2$$

where the angles δ_1 and δ_2 are in radians.

The area of parcel C-PT-B-D can also be determined in a similar fashion.

$$\begin{aligned} A_{C-PT-B-D} &= A_{O-C-PT} - A_{O-C-D} - A_{O-B-D} \\ &= \frac{1}{2} R_1^2 \left(\Delta - \delta_1 \right) - \frac{1}{2} R_1 R_2 \sin \angle_{COB} - \frac{1}{2} R_2^2 \left(\Delta - \delta_2 \right) \end{aligned}$$

Danial shows that the angles between the chords and the sides can be calculated from the following relationships. For parcel PC-C-D-A:

$$\angle_{PC-A-D} = 90^{o} + \delta_2 / 2$$

$$\angle_{A-PC-C} = 90^{o} - \delta_1 / 2$$

$$\angle_{PC-C-D} = 90^{o} - \delta_1 / 2 + \alpha$$

$$\angle_{C-D-A} = 90^{o} - \alpha + \delta_1 - \delta_2 / 2$$

For parcel C-PT-B-D:

$$\angle_{C-PT-B} = 90^{o} - \frac{\Delta - \delta_1}{2}$$

$$\angle_{PT-B-D} = 90^{o} + \frac{\Delta - \delta_2}{2}$$

$$\angle_{PT-C-D} = 90^o - \frac{\Delta - \delta_1}{2} - \alpha$$

$$\angle_{C-D-B} = 90^o + \alpha - \delta_1 + \frac{\Delta + \delta_2}{2}$$

The second situation for Case 1 is where the dividing line is parallel to one of the radial lines. Line CD is located at a distance w from the line PT-B (figure 19).

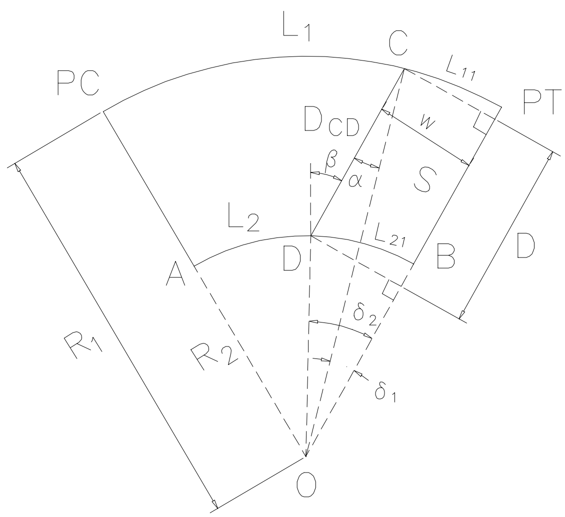

Figure 19. Area partitioning of pie-shaped lot by a line parallel to one of the radial lines.

Then,

$$\delta_1 = \sin^{-1}\left(\frac{w}{R_1}\right)$$

$$\delta_2 = \sin^{-1}\left(\frac{w}{R_2}\right)$$

Because line CD is parallel to PT-B, then $\alpha = \delta_1$ and $\beta = \delta_2$. The arc length L_{11} is found using

$$L_{11} = R_1\delta_1$$

while the arc length L_{21} becomes

$$L_{21} = R_2\delta_2$$

where δ_1 and δ_2 are in radians. Recognizing that $\angle_{CDO} = 180^\circ - [\alpha + (\delta_2 - \delta_1)]$, then the cosine law can be used to compute the distance D_{CD}.

$$D_{CD} = R_1^2 + R_2^2 - 2R_1R_2 \cos\angle_{DOC}$$

This distance can also be computed as

$$D_{CD} = \sqrt{R_1^2 - w^2} - \sqrt{R_2^2 - w^2}$$

The area of parcel PC-C-D-A is found to be

$$\begin{aligned} A_{PC-C-D-A} &= A_{PC-C-O} - A_{C-D-O} - A_{A-D-O} \\ &= \frac{1}{2}R_1^2(\Delta - \delta_1) - \frac{1}{2}R_1R_2 \sin\angle_{DOC} - \frac{1}{2}R_2^2(\Delta - \delta_2) \end{aligned}$$

where Δ, δ_1, and δ_2 are in radians. The area of parcel C-PT-B-D becomes

$$\begin{aligned} A_{C-PT-B-D} &= A_{C-PT-O} - A_{C-O-D} - A_{D-B-O} \\ &= \frac{1}{2}R_1^2\delta_1 - \frac{1}{2}R_1R_2 \sin\angle_{DOC} - \frac{1}{2}R_2^2\delta_2 \end{aligned}$$

Danial (1990) identifies Case 2 as being the division here one side is not a radial line (figure 10). Line AB within polygon ABCD is not a radial line. The given data are R_1, R_2, L_1, and L_2. From basic curve geometry, the central angle to the two curves can be shown as, in radians:

$$\Delta_1 = \frac{L_1}{R_1} \qquad \Delta_2 = \frac{L_2}{R_2}$$

Using the cosine law, the distance from A to B is

$$S_1 = \sqrt{R_1^2 + R_2^2 - 2R_1R_2 \cos\delta}$$

The angle at B between A and the center of the arc is computed from the sine law.

$$\alpha = \sin^{-1}\left(\frac{R_2}{R_1}\sin\delta\right)$$

The angle at A then becomes

$$\angle_{BAO} = 180^{o} - (\alpha + \delta)$$

The other angles in polygon AQBCD are then

$$\angle_{ADC} = 90^o + \Delta_2/2$$
$$\angle_{DCB} = 90^o - \Delta_1/2$$
$$\angle_{CBA} = 90^o - \Delta_1/2 + \alpha$$
$$\angle_{BAD} = 90^o - \Delta_2/2 - \alpha + \Delta_1$$

The area of polygon ABCD is then defined as

$$\begin{aligned} A_{ABCD} &= A_{S_{BOC}} + A_{T_{BAO}} - A_{S_{AOD}} \\ &= \frac{1}{2} R_1^2 \Delta_1 + \frac{1}{2} R_1 R_2 \sin \delta - \frac{1}{2} R_2^2 \Delta_2 \end{aligned}$$

The subscript S indicates the area of a sector while the subscript T delineates the area of a triangle. Δ_1 and Δ_2 are in radians.

The second case presented by Danial (1990) involves constructing a dividing line that is parallel to a non-radial side of the pie-shaped parcel (figure 20). The known quantities are the two radii, R_1 and R_2, the two arc lengths, L_1 and L_2, the distance along the non-radial side of the polygon, $S_1 = D_{AB}$, and the offset distance from the non-radial side to the dividing line, w.

The angle at A from B to the center of the curve, O, is

$$\angle_{BAO} = 180^o - (\alpha_1 + \delta_1)$$

From this,

$$\beta = \alpha_1 + \delta_1$$

There are two possible solutions to this general problem of dividing a pie-shaped polygon. Figure 20 depicts the situation where Δ_1 is smaller than Δ_2. This case will be discussed first.

Draw a line from A to a point, G, on line EF, or its extension, that is perpendicular to that line. The distance of line AG = w. Line AH is the tangent to the second or lower curve. The angle between this tangent and line AG is defined as β. Designating δ as the central angle subtending the arc from AF, then the angle from the chord AF to the line AG is

$$\angle_{FAG} = \vartheta = \beta - \frac{\delta}{2}$$

Then, the offset distance, w, can be defined trigonometrically as

$$w = D_{AF}\cos\vartheta = 2R_2\sin\left(\delta/2\right)\cos\vartheta$$

Substitute the value for ϑ gives

$$w = 2R_2\sin\left(\frac{\delta}{2}\right)\cos\left(\beta - \frac{\delta}{2}\right)$$

Solve for δ by first rearranging this formula.

$$\frac{w}{2R_2} = \sin\left(\frac{\delta}{2}\right)\cos\left(\beta - \frac{\delta}{2}\right)$$

Using the trigonometric function of a difference in two angles and rearranging gives

$$\frac{w}{2R_2} = \cos\beta\sin\left(\delta/2\right)\cos\left(\delta/2\right) + \sin\beta\sin^2\left(\delta/2\right)$$

From the trigonometric function of half angles, the equation becomes

$$\frac{w}{2R_2} = \cos\beta\left(\frac{1-\cos\delta}{2}\right)^{1/2}\left(\frac{1+\cos\delta}{2}\right)^{1/2} + \sin\beta\left(\frac{1-\cos\delta}{2}\right)$$

Reducing yields,

$$\frac{w}{R_2} = \cos\beta\sin\delta + \sin\beta - \sin\beta\cos\delta$$

Recognize the double angle formula gives

$$\frac{w}{R_2} - \sin\beta = \sin(\delta - \beta)$$

From this, we can see that

$$\delta = \beta + \sin^{-1}\left(\frac{w}{R_2} - \sin\beta\right)$$

where δ and β are in radians.

With δ known, ϑ can now be computed. The chord distance between A and F is computed as

$$D_{AF} = R_2 \delta$$

where δ is in radians. The angle at F from A to G is

$$\angle_{AFG} = 180^o - \left(90^o + \vartheta\right) = 90^o - \vartheta$$

The complement at F from A to E becomes $180^o - 90^o + \vartheta = 90^0 + \vartheta$. The angle at F from A to O is

$$\angle_{AFO} = 180^o - \left(90^o - \beta + \vartheta + \delta\right) = 90^o + \beta - \vartheta - \delta$$

From these relationships, the angle at F can be represented as

$$\begin{aligned} \angle_{OFE} &= \angle_{AFO} + \angle_{AFE} \\ &= 180^o + \beta - \delta \end{aligned}$$

Using the law of sines for triangle OEF, the angle α_2 is found using

$$\alpha_2 = \sin^{-1}\left(\frac{R_2}{R_1} \sin \angle_{OFE}\right)$$

The angle δ_2 in triangle OEF is

$$\begin{aligned} \delta_2 &= 180^o - \left(\alpha_2 + \angle_{OFE}\right) = 180^o - \alpha_2 - 180^o - \beta + \delta \\ &= \alpha_2 - \beta + \delta \end{aligned}$$

The angle at the center of the arc from B to E, designated as δ_3, is

$$\begin{aligned} \delta_3 &= \delta - \left(\delta_1 + \delta_2\right) \\ &= -\delta_1 + \beta + \alpha_2 \end{aligned}$$

Recall that $\beta = \alpha_1 + \delta_1$. Substitute this into the equation for δ_3 gives us

$$\delta_3 = \alpha_1 + \alpha_2$$

With the central angle between B and E known, the arc length is found by multiplying δ_3, in radians, by the radius.

$$L_{BE} = R_1 \delta_3$$

Danial (1990) gives the angles for the two parcels. For polygon ABEF, the following angles are defined:

$$\angle_{BEF} = 90^o - \vartheta$$

$$\angle_{ABE} = 90^o - \frac{\delta_2}{2} + \alpha_1$$

$$\angle_{BEF} = 90^o - \frac{\delta_3}{2} + \alpha_2$$

$$\angle_{EFA} = 90^o + \vartheta$$

For polygon CDFE, the angles are:

$$\angle_{ECD} = 90^o - \frac{\Delta_1 - \delta_3}{2}$$

$$\angle_{CDF} = 90^o + \frac{\Delta_2 - \delta}{2}$$

$$\angle_{DFE} = 90^o - \frac{\Delta_1 - \delta_3}{2} - \alpha_2$$

$$\angle_{FEC} = 90_o + \frac{\Delta_2 - \delta}{2} + \alpha_2 + \delta_2$$

The area for both polygons are shown as follows.

$$A_{ABEF} = A_{S_{BOE}} + A_{T_{AOB}} + A_{T_{EOF}} - A_{S_{AOF}}$$

$$= \frac{R_1^2 \delta_3}{2} + \frac{R_1 R_2 \sin \delta_1}{2} + \frac{R_1 R_2 \sin \delta_2}{2} - \frac{R_1^2 \delta}{2}$$

where the subscript S indicated the area of a sector and the subscript T identifies the area of a triangle. For parcel CDFE the area is

$$A_{CDFE} = A_{S_{EOC}} - A_{T_{EOF}} - A_{S_{FOD}}$$

$$= \frac{R_1^2 (\Delta_1 - \delta_3)}{2} - \frac{R_1 R_2 \sin \delta_2}{2} - \frac{R_2^2 (\Delta_2 - \delta)}{2}$$

Case 2 also can take on a different configuration where Δ_1 is greater than Δ_2. This is depicted in figure 21. Using the same approach as shown in the situation where $\Delta_1 < \Delta_2$, the angle δ can be shown to be

$$\delta = -\beta + \sin^{-1}\left(\frac{w}{R_2} + \sin\beta\right)$$

Danial also shows that the interior angles of polygon ABEF can be shown as

$$\angle_{BEF} = 90^o - \vartheta$$
$$\angle_{ABE} = 90^o - \frac{\delta_2}{2} - \alpha_1$$
$$\angle_{BEF} = 90^o - \frac{\delta_3}{2} + \alpha_2$$
$$\angle_{EFA} = 90^o + \vartheta$$

For polygon CDFE the angles are the same as presented before. The angle at F is

$$\angle_{EFO} = 180^o - \beta - \delta$$

The area of both polygons can be computed as before.

$$A_{ABEF} = \frac{R_1^2\delta_3}{2} + \frac{R_1R_2\sin\delta_1}{2} + \frac{R_1R_2\sin\delta_2}{2} - \frac{R_1^2\delta}{2}$$

$$A_{CDFE} = \frac{R_1^2(\Delta_1 - \delta_3)}{2} - \frac{R_1R_2\sin\delta_2}{2} - \frac{R_2^2(\Delta_2 - \delta)}{2}$$

There is a second solution to this problem as Danial points out. Draw a line parallel to the non-radial side (figures 20 and 21) at a distance w_1. This offset distance is shown to be

$$w_1 = R_1 \sin\alpha_1$$

When Δ_1 is smaller than Δ_2, the distance from this new line to the dividing line, w_2, is

$$w_2 = w - R_1 \sin\alpha_1$$

and the distance of the dividing line becomes

$$S_D = \sqrt{R_1^2 - w_2^2} - \sqrt{R_2^2 - w_2^2}$$

From triangle EOF, we can see that

$$\delta_3 = \alpha_1 + \alpha_2$$

and

$$\delta = \delta_1 + \delta_2 + \delta_3$$

When Δ_1 is greater than Δ_2 then,

$$w_2 = w + R_1 \sin \alpha_1$$

$$\delta_3 = -\alpha_1 + \alpha_2$$

$$\delta = -\delta_1 + \delta_2 + \delta_3$$

One method of dividing a four-sided area where one side is a circular curve was presented by Easa [1992]. In figure 20, OJ is a radial line running parallel with AD. It is desired to divide parcel ADFE in such a manner that the side EF is parallel to AD. From the geometry, the angle α is computed as

$$\alpha = \lambda\left(\angle_{BAD} - 90^{o}\right) + \frac{\Delta}{2}$$

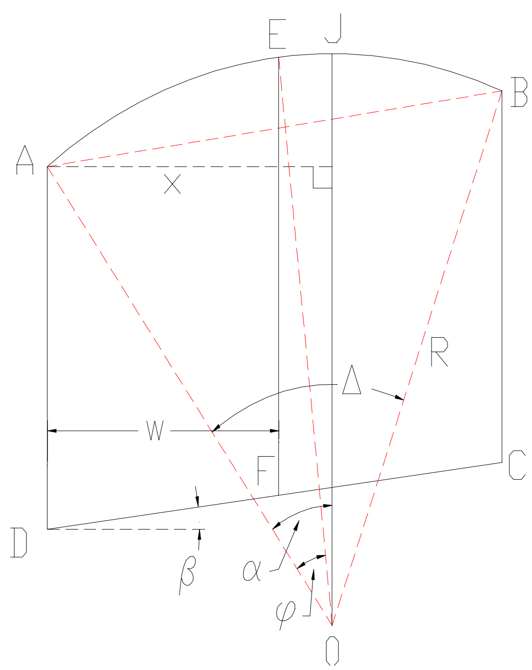

Figure 20. Geometry of an area partitioning problem with one side being a curved line.

where λ is a constant that is equal to 1 for convex circular curves and –1 for concave circular curves. The distance x between OJ and AD is found from trigonometry as

$$x = R \sin\alpha$$

The angle β is found using the simple relationship

$$\beta = \angle_{ADC} - 90^o$$

Finally, the distance EF is found from

$$D_{EF} = D_{AD} - \lambda R \cos\alpha + \lambda\sqrt{R^2 - (x - w)^2} + w \tan\beta$$

where w is the offset distance given between EF and AD.

In the previous discussion the offset distance was given. Much more frequently, one needs to divide the parcel using a specified area. Easa [1992] presents an iterative approach for four different situations: Case 1 where EF is parallel to AD, Case 2 where EF is not parallel to AD, Case 3 where the line dividing the parcel is fixed at F, and Case 4 where the dividing line is fixed at E.

CASE 1

From basic geometry we know that the area of the figure OAE using the arc distance AE_{arc} is

$$A = \frac{\varphi}{2}\pi R^2$$

where φ is the central angle for the arc. The area of the same figure using the chord distance AE_{chord} is

$$A = \frac{R^2}{2}\sin\varphi$$

The area of the segment between the arc and the chord is the difference between these two areas

$$A_{Seg} = \frac{R^2}{2}\left(\frac{\varphi\pi}{180^o} - \sin\varphi\right)$$

The central angle is

$$\varphi = 2\sin^{-1}\left[\frac{D_{AE_{arc}}}{2R}\right]$$

The chord distance AE_{chord} is computed as

$$D_{AE_{chord}} = \left[\left(D_{EF} - D_{AD} - w \tan\beta \right)^2 + w^2 \right]^{1/2}$$

From these developments, the area (A) of the figure ADFE is computed using the following relationship.

$$A = \frac{w}{2}\left(D_{AD} + D_{EF} \right) + \lambda A_{Seg}$$

Solving for w yields

$$w = \frac{2\left(A - \lambda A_{Seg} \right)}{\left(D_{AD} + D_{EF} \right)}$$

The length of the dividing line (D_{EF}) is then computed using the formula above. To solve this problem, an initial guess for w is inserted into the equation for the D_{EF}. Then, using the last equation for w, a better estimate of w is computed. This process is iterated until the computed value of w from (11) does not deviate by more than some criteria (i.e., 0.001') from the value of w used in computing D_{EF}. Easa [1992] gives the steps as follows:

1. Compute α, x, and β.
2. Estimate a value for w.
3. Compute the distance of the dividing line, D_{EF} .
4. Compute the chord distance, AE_{chord} .
5. Compute φ and A_{Seg}.
6. Compute a new value for w. Compare it to the current estimate of w.
7. Repeat steps 3-6 until the difference between the current estimate of w and the computed value is within the tolerance established for the calculation.

CASE 2

Figure 21 shows the geometry when the dividing line (EF) is not parallel to AD. A line AD' is constructed parallel to the dividing line EF. The angle between AD and AD' is referred to as the offset angle and is designated as δ. For the figure AEFD', use the same approach as presented for Case 1. Angle BAD is equal to $90^o - \Delta/2$. What remains to be done is the computation of the area ADD'. Using the sine law, the distances $D_{DD'}$ and $D_{AD'}$ are computed as follows:

$$D_{DD'} = \frac{D_{AD} \sin\delta}{\sin\angle_{AD'C}}$$

$$D_{AD'} = \frac{D_{AD} \sin\angle_{ADC}}{\sin\angle_{AD'C}}$$

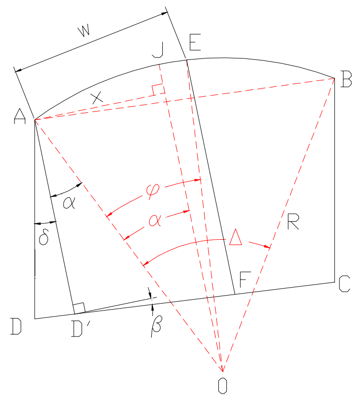

Figure 21. Geometry of area partitioning for Case 2.

The area is computed as:

$$A_{ADD'} = \frac{1}{2} D_{AD'} D_{AD} \sin\delta$$

The total area used in the calculations is the area from AEED' plus the area ADD' if δ lies to the left of AD (minus if on the right). The process is iterated as before.

CASE 3

In Case 3 the dividing line is fixed through point F (distance DF is given). It is also assumed that the other end of the dividing line lies on the arc AB_{Arc}. The lines AD' and OJ are constructed to be parallel to the dividing line EF. The direction of the dividing line (δ) is unknown. From figure 3, one can see that

$$\angle_{BAD'} = \angle_{BAD} + \delta$$

$$\angle_{AD'C} = \angle_{ADC} - \delta$$

Using the same relationships presented earlier dealing with α, x, and β, one can develop the following:

$$\alpha = \lambda\left(\angle_{BAD'} - 90^{o}\right) + \frac{\Delta}{2}$$

$$x = R\sin\alpha$$

$$\beta = \angle_{AD'C} - 90^{o}$$

The offset distance (w) is

$$w = \left(D_{DD'} + D_{DF}\right)\sin\angle_{AD'C}$$

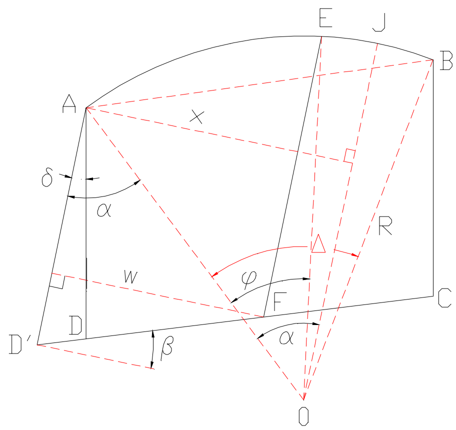

Figure 22. Geometry for area partitioning for Case 3.

The length of the dividing line is found using a modified version of formula to compute D_{EF} presented earlier.

$$D_{EF} = D_{AD'} - \lambda R\cos\alpha + \lambda\sqrt{R^2 - (x-w)^2} + w\tan\beta$$

The area of the segment between the chord and the arc is found using the formula for computing the area of the segment between the arc and the chord (A_{Seg}). The area relationship is

$$A = \frac{w}{2}\left(D_{AD'} + D_{EF}\right) + \lambda A_{Seg} - A_{ADD'}$$

The direction of the dividing line ($\angle_{EFC}$) is the same as $\angle_{AD'C}$ from which one can state

$$\angle_{EFC} = \angle_{ADC} - \delta$$

Easa [1992]] points out that the unknown is δ and this must be solved for iteratively by using the area relationship

$$f(\delta) = A - \frac{w}{2}\left(D_{AD'} + D_{EF}\right) - \lambda A_{Seg} + A_{ADO} = 0$$

To solve this problem, two values for δ must be selected (δ_1 and δ_2) such that the corresponding $f(\delta_1)$ and $f(\delta_2)$ from the relationships for f(δ) have opposite signs. Then

$$\delta_3 = \delta_2 - f(\delta_2)\frac{\delta_2 - \delta_1}{f(\delta_2) - f(\delta_1)}$$

A new value for δ_1 is then found using the test: if $f(\delta_3)$ has the opposite sign of $f(\delta_1)$ then $\delta_1 = \delta_3$. This process is continued until $|\delta_2 - \delta_1| \le \varepsilon_1$ and $|f(\delta_3)| \le \varepsilon_2$ where ε_1 and ε_2 are the tolerances established for this problem. Easa [1992] identifies the steps necessary for the solution of the Case 3 problem:

1. Compute α, x, and β.
2. Find $D_{DD'}$ and $D_{AD'}$.
3. Compute $A_{ADD'}$.
4. Calculate w.
5. Find D_{EF}.
6. The chord distance AE_{Chord} is computed using $D_{AD'}$ instead of D_{AD}.
7. Determine φ.
8. Determine the area A_{Seg}.
9. Calculate $f(\delta)$.

Steps 1-9 are performed first with $\delta = \delta_1$ and then with $\delta = \delta_2$. The equation for δ_3 is then applied and the appropriate changes are made to either δ_1 or δ_2 and the process is repeated until the solution converges.

CASE 4

This case is very simple because the point is fixed on the curve. Because of that, the distances D_{EF} and D_{DF} can be easily calculated.

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